REDUCED KEPLER PROBLEM IN ELLIPTIC COORDINATES

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March 1999

Introduction. The "2-body problem with central interaction" can be described

$$\mathsf{L} = \tfrac{1}{2}m_1\dot{\boldsymbol{x}}_1\cdot\dot{\boldsymbol{x}}_1 + \tfrac{1}{2}m_2\dot{\boldsymbol{x}}_2\cdot\dot{\boldsymbol{x}}_2 - U(r)$$
$$r \equiv \text{length of } \boldsymbol{r} \equiv \boldsymbol{x}_2 - \boldsymbol{x}_1$$

but by familiar reduction—write

$$\left.\begin{array}{l} \boldsymbol{x}_1 = \boldsymbol{X} + \boldsymbol{r}_1 \\ \boldsymbol{x}_2 = \boldsymbol{X} + \boldsymbol{r}_2 \end{array}\right\} \text{and require } m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = \boldsymbol{0}\text{, giving } \left\{\begin{array}{l} \boldsymbol{x}_1 = \boldsymbol{X} - \frac{m_2}{m_1+m_2}\boldsymbol{r} \\ \boldsymbol{x}_2 = \boldsymbol{X} + \frac{m_1}{m_1+m_2}\boldsymbol{r} \end{array}\right.$$

—assumes a form

$$\mathsf{L} = \tfrac{1}{2}M\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{X}} + \left\{\tfrac{1}{2}m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - U(r)\right\} \text{ with } \left\{\begin{array}{ll} M \equiv m_1 + m_2 & : \text{ total mass} \\ m \equiv \frac{m_1m_2}{m_1+m_2} & : \text{ reduced mass} \end{array}\right.$$

which can be considered to describe

- motion $\boldsymbol{r}(t)$ of a fictitious "reduced mass" m in a central force field pinned at the center of mass, superimposed upon
- unaccelerated drift $\boldsymbol{X}(t)$ of the center of mass.

One is led thus from a physical 2-body problem to an abstractly equivalent one-body problem—the "reduced central force problem"

$$L = \tfrac{1}{2}m\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - U(r) \tag{1}$$

which in the case $U(r) = -kr^{-1}$ becomes the "reduced Kepler problem." Further reduction is made possible by the observation that, because the force is central, the orbit $\boldsymbol{r}(t)$ is *confined necessarily to a central plane* (i.e., to a plane that intersects the origin). That fact can be obtained as a corollary of a more restrictive condition; namely, that (whether one argues by Noether's theorem

from the rotational symmetry of (1), or from evaluation of the relevant Poisson brackets) *angular momentum is conserved*:

$$\dot{\boldsymbol{L}} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{L} \equiv \boldsymbol{r}\times\boldsymbol{p}$$
$$\boldsymbol{p} \equiv \partial L/\partial\dot{\boldsymbol{r}} = m\dot{\boldsymbol{r}}$$

Evidently $\boldsymbol{L}$ stands normal to the orbital plane.

The 3-body problem (with gravitational interaction) is well-known to be analytically intractable except in several classes of special cases, among which is the "problem of two centers," first studied by Leonard Euler. Euler imagined two of the masses (M_1 and M_2) to be pinned (and their coordinates therefore to be removed from the list of dynamical variables; the 3-body problem has become at this point a one-body problem). The orbit of the third mass m is then generally *not* confined to a plane, but—since m never experiences force components normal to the (M_1, M_2, m)-plane—will be so confined in $\boldsymbol{p}_{\text{initial}}$ lies in that plane. It is to this case that Euler restricted his attention.

In a recent essay[1] I had occasion to review, and in some respects to extend, Euler's solution of the problem just described. Results appropriate to the Kepler problem were obtained there by a several distinct limiting procedures. But the density of the argument was so great as frequently to obscure the significance of the results obtained. Here I will attempt to achieve a more focused account of the Keplerean significance of Euler's method by eliminating all explicit reference to Euler's "second force center." I will, of course, have things to say about the familiar "orbital" aspects of the Kepler problem, but will have special interest in those aspects of its classical physics which relate more directly to the associated quantum theory; orbital notions contributed centrally to Bohr's account of the physics of the hydrogen atom, but in the line of development which proceeded historically from Bohr to Schrödinger such notions became progressively more subordinate to ideas borrowed from Hamilton-Jacobi theory.

1. Reduced central force problem in polar coordinates. Erect—with origin at the (inertial) force center—a Cartesian frame with (as a matter of convenience) z-axis aligned with $\boldsymbol{L}$, and write

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}$$

Then

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U\big(\sqrt{x^2+y^2}\,\big) \tag{2.1}$$
$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{2.2}$$

[1] "Kepler problem by descent from the Euler problem" (1996).

give

$$\left.\begin{aligned} m\ddot{x} + \frac{x}{\sqrt{x^2+y^2}} \cdot U'\big(\sqrt{x^2+y^2}\,\big) = 0 \\ m\ddot{y} + \frac{y}{\sqrt{x^2+y^2}} \cdot U'\big(\sqrt{x^2+y^2}\,\big) = 0 \end{aligned}\right\} \tag{3.1}$$

and

$$\left.\begin{aligned} m\ddot{r} + U'(r) - mr\dot{\theta}^2 = 0 \\ \tfrac{d}{dt}\big[mr^2\dot{\theta}\,\big] = 0 \end{aligned}\right\} \tag{3.2}$$

respectively. Equations (3.1) are coupled *except in the case* $U(r) = kr^2$, which is the case of a 2-dimensional harmonic oscillator; m is bound to the force center by a "spring." Equations (3.2) are coupled in *every* case, but in every case one has

$$mr^2\dot{\theta} = \text{constant with dimension of angular momentum, call it } \ell \tag{4}$$

giving

$$m\ddot{r} + U'(r) - \ell^2/mr^3 = 0 \tag{5}$$

from which all reference to θ has disappeared. Notice that we have achieved (5) not as an artifact of decoupling (of the sort exhibited by (3.1) in the harmonic case) but by appeal to a conservation law; this is a circumstance recalled by the ℓ^2 in (5). The conservation law in question can be expressed

$$\begin{aligned} \dot{p}_\theta = 0 \quad \text{with} \quad p_\theta \equiv \tfrac{\partial}{\partial\dot{\theta}} L = mr^2\dot{\theta} \;:\; &\text{momentum conjugate to } \theta \\ &= xp_y - yp_x \\ &= (\boldsymbol{x}\times\boldsymbol{p})_z \end{aligned} \tag{6}$$

and arises because θ does not appear among the arguments of the Lagrangian; the polar coordinate system acquires its "universal pertinence" from the fact that it is adapted to the rotational symmetry of the central force problem (promotes the argument of $U(\bullet)$ to the status of a coordinate).

Further progress is made by appeal to energy conservation—as yet unexploited. From

$$\begin{aligned} E &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \\ &= \tfrac{1}{2}m\dot{r}^2 + U(r) + \tfrac{1}{2}\ell^2/mr^2 \end{aligned} \tag{7}$$

we obtain

$$\tfrac{dr}{dt} = \sqrt{\tfrac{2}{m}\left[E - U(r) - \tfrac{\ell^2}{2mr^2}\right]} \tag{8}$$

at which point the dynamical problem has been "reduced to quadrature." Further progress hinges upon one's ability to

- evaluate an integral;
- execute a functional inversion;
- evaluate another integral.

It would be pointless for me to pursue the details; they have been burnished over the ages, and are well-described in (for example) Chapter 3 of Goldstein.[2] I will confine my remarks to a few relatively non-standard points.

In dynamics generally—not just in connection with the 2-body problem—it is possible (and sometimes useful) to partition the "dynamical problem" (describe $\boldsymbol{x}(t)$) into two parts:

- construct a parameterized description $\boldsymbol{x}(\lambda)$ of the trajectory;
- describe temporal progress $\lambda(t)$ along that trajectory.

The former problem is, within the present context, usually interpreted as an assignment to construct $r(\theta)$, and is approached as follows:[3] divide this variant of (4)

$$\tfrac{d\theta}{dt} = \ell/mr^2$$

into (8) and obtain

$$\tfrac{dr}{d\theta} = \tfrac{mr^2}{\ell}\sqrt{\tfrac{2}{m}\left[E - U(r) - \tfrac{\ell^2}{2mr^2}\right]} \tag{9}$$

No θ appears on the right, so the problem has again been reduced to quadrature and a functional inversion—which, when they can be performed, yield $r(\theta; E, \ell)$. I digress now to describe an alternative derivation of (9).

Write $r(\theta)$ to describe a plane curve linking point $\{r_1, \theta_1\}$ to point $\{r_2, \theta_2\}$. The Euclidean length of such a curve $\mathcal{C}$ can be described

$$\int_{\theta_1}^{\theta_2} \sqrt{\mathring{r}^{\,2} + r^2}\, d\theta$$

where $\mathring{r} \equiv dr/d\theta$. The mechanical analog of the "optical path length" of $\mathcal{C}$ is an attribute of $\mathcal{C}$ that becomes meaningful when we imagine the curve to have been traced with conserved energy E by a particle of mass m, and is given by[4]

$$\left.\begin{aligned} A[r(t)] &\equiv \int_{\theta_1}^{\theta_2} \frac{1}{n(r;E)}\sqrt{\mathring{r}^{\,2} + r^2}\, d\theta \\ \frac{1}{n(r;E)} &\equiv \sqrt{\tfrac{2}{m}\big[E - U(r)\big]} \end{aligned}\right\} \tag{10}$$

Jacobi's principle (the mechanical analog of Fermat's principle of least time, sometimes known as the "principle of least action") asserts that the *physical* trajectory

$$\{r_1, \theta_1\} \xrightarrow[\text{isoenergetic trajectory of particle with mass } m]{} \{r_2, \theta_2\}$$

[2] H. Goldstein, *Classical Mechanics* (2nd edition, 1980). All future references to "Goldstein" will be to this classic text.

[3] See §3–5 in Goldstein.

[4] See §5 in "Geometrical mechanics: Remarks commemorative of Heinrich Hertz" (1994).

is the trajectory which *extremizes* $A[r(\theta)]$:

$$\delta A[r(\theta)] = 0 \quad \Longrightarrow \quad \left\{ \frac{d}{d\theta} \frac{\partial}{\partial \mathring{r}} - \frac{\partial}{\partial r} \right\} \mathcal{A}_E(r, \mathring{r}) = 0 \tag{11}$$

$$\mathcal{A}_E(r, \mathring{r}) \equiv \sqrt{\tfrac{2}{m}\big[E - U(r)\big]\big[\mathring{r}^{\,2} + r^2\big]}$$

One could—with patience, on a large sheet of paper—actually write out the second order differential equation

$$G(r, \mathring{r}, \mathring{\mathring{r}}) = 0 \ : \ \text{mechanical analog of the optical "ray equation"}$$

of which (11) speaks, but the result (I am informed by *Mathematica*) is a mess: that, evidently, is not the way to go; some circumspection is called for. I proceed in geometrical/non-temporal mimicry of a line of argument standard to temporal mechanics.

Lagrangian dynamics supplies the general proposition that if $L(q, \dot{q})$ does not depend explicitly on t then $J(q, \dot{q}) \equiv L - \dot{q} \cdot (\partial L / \partial \dot{q})$ is a constant of motion:

$$\tfrac{\partial}{\partial t} L = 0 \quad \Longrightarrow \quad J(q, \dot{q}) = \text{constant}$$

$J(q, \dot{q})$ is "Jacobi's integral"—interpretable as "total energy" $T + U$ in its most commonly-encountered manifestations, and recommended to our attention by Noether's theorem as the object of interest whenever time-translation is a map of interest. All of which has much to do with the abstract calculus of variations, and only incidentally to do—by way of "illustrative application"—with dynamics. Looking in this light back now to (11), we observe that $\mathcal{A}_E(r, \mathring{r})$ displays no explicit dependence upon the independent variable θ: $\frac{\partial}{\partial \theta} \mathcal{A}_E = 0$. The implication is that

$$\mathcal{J}(r, \mathring{r}) \equiv \mathcal{A}_E - \mathring{r} \cdot (\partial \mathcal{A}_E / \partial \mathring{r}) \tag{12.1}$$

has the property that $\frac{\partial}{\partial \theta} \mathcal{J}(r, \mathring{r})$ vanishes on every solution $r(\theta)$ of (11):

$$\mathcal{J}(r, \mathring{r}) \text{ is } \textit{constant on every trajectory} \tag{12.2}$$

The argument culminating in (12) is of some general interest: it alerts us to the fact that *Noether's theorem possesses a domain of applicability which extends far beyond the dynamical domain* with which we physicists are most familiar.[5] But concentrating now on the particulars of the situation, we have

$$\begin{aligned} \mathcal{J} &= \sqrt{\tfrac{2}{m}\big[E - U\big]\big[\mathring{r}^{\,2} + r^2\big]} - \mathring{r} \cdot \frac{\tfrac{2}{m}\big[E - U\big]\mathring{r}}{\sqrt{\tfrac{2}{m}\big[E - U\big]\big[\mathring{r}^{\,2} + r^2\big]}} \\ &= \frac{\tfrac{2}{m}\big[E - U\big]\big[\mathring{r}^{\,2} + r^2 - \mathring{r}^{\,2}\big]}{\sqrt{\tfrac{2}{m}\big[E - U\big]\big[\mathring{r}^{\,2} + r^2\big]}} \end{aligned}$$

[5] I suspect that the non-dynamical (geometrical) manifestations of her train of thought were known long before Noether herself entered the picture, and that she was familiar with them, but I have not had opportunity to discover whether the historical literature supports my suspicion.

giving

$$\mathcal{J}(r,\,\mathring{r}\,) = r^2\sqrt{\frac{E-U(r)}{\mathring{r}^{\,2}+r^2}} = \text{constant, call it } \sqrt{\ell^2/2m} \tag{13}$$

This is precisely equivalent to (9). Recent discussion has taught us nothing we did not already know about orbital geometry in central force problems, but it is interesting to see that (and how) the variational principle (11)—which pertains in principle to a much broader class of problems—can be made to do useful work. And the discussion has served to alert us to the *geometrical potentialities of Noether's theorem*.

The orbital equation (13) can be written

$$\begin{aligned}
\mathring{r}^{\,2} &= r^2\left\{\frac{E-U(r)}{\ell^2/2mr^2} - 1\right\} \\
&= (2mr^4/\ell^2)\Big\{E - U(r) - \ell^2/2mr^2\Big\} &(14.1)\\
&= (2mr^4/\ell^2)\Big\{E - U_{\text{eff}}(r;\ell^2)\Big\} &(14.2)
\end{aligned}$$

$$U_{\text{eff}}(r;\ell^2) \equiv \tfrac{\ell^2}{2mr^2} + U(r) \tag{15}$$

A change of variable $r \mapsto z \equiv 1/r$ (which entails $\mathring{r} \mapsto \mathring{z} = -\mathring{r}/r^2$) permits this result to be cast into a form

$$\tfrac{1}{2}m\mathring{z}^{\,2} = \mathcal{E} - V(z) \quad \text{with} \quad \begin{cases} \mathcal{E} \equiv (m/\ell)^2 E \\ V(z) \equiv (m/\ell)^2 U_{\text{eff}}(z^{-1};\ell^2) \end{cases}$$

which makes even more obvious the fact that (14) presents a problem which is abstractly equivalent to the problem of a particle moving one-dimensionally in a potential well; the role of time t has been taken over now by a *geometrical* variable θ, and z has the dimension of *reciprocal* length, but—those distinctions notwithstanding—the imagery of the latter problem can be borrowed intact: orbits are excluded from regions where $\mathcal{E} - V(z) < 0$, are confined to the interior of annular regions bounded by the analog of "turning points," become circular at the analog of stable/unstable "equilibrium points." Orbits passing through a point $r(\theta)$ are

- destined to collapse ($r \to 0$)
- bounded (r is oscillatory, a periodic function of θ)
- destined to evaporate ($r \to \infty$)

according to the placement of the nearest turning points, and that placement depends upon the values which have been assigned to the physical parameters E and ℓ^2 (which can usefully be considered to mark a point on a semi-infinite "control plane"). Typical aspects of the situation are illustrated in Figure 1.

In view of my destination, and to eliminate distracting complications in the shorter term, I restrict my attention now and henceforth to the power-law potentials

$$U(r) = kr^n \tag{16}$$

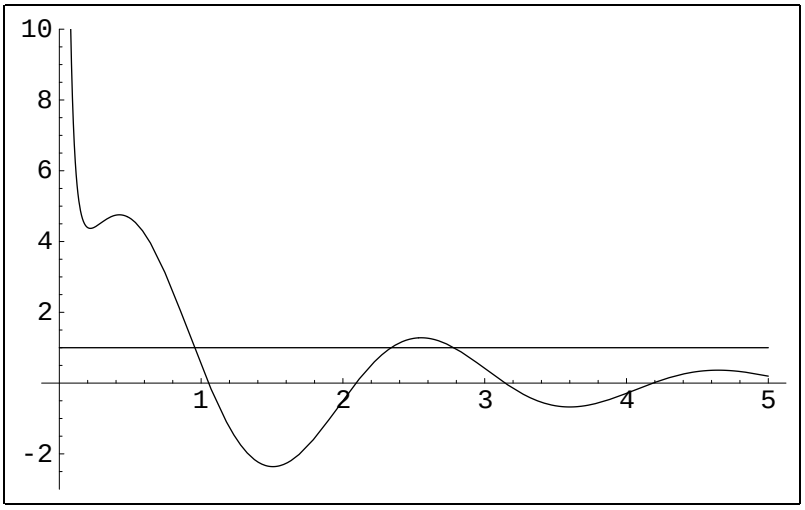


FIGURE 1: *I have, for illustrative purposes, assigned the potential energy function the physically artificial structure*

$$U(r) = ae^{-br}\sin cr$$

and plotted $U_{\text{eff}}(r,\ell^2)$*. From such figures one can, for each assigned value of* E*, read off orbital turning point data. Here identical values of* E *and* ℓ^2 *support both*

$$\text{a bound orbit with } 0.9 \leqslant r \leqslant 2.3, \text{ and}$$
$$\text{an unbound orbit with } 2.8 \leqslant r$$

Analogous figures arise from each assigned value of ℓ^2*. For the* ℓ^2*-value shown, no orbit is possible if* $E < 2.2$*. The spike on the left arises from the* $\ell^2/2mr^2$*-term which enters additively into the definition of* $U_{\text{eff}}(r,\ell^2)$*.*

and will give special attention to the

$$\text{HARMONIC POTENTIAL:} \quad U(r) = +kr^2$$
$$\text{ATTRACTIVE COULOMB POTENTIAL (KEPLER PROBLEM):} \quad U(r) = -kr^{-1}$$

where the "strength parameter" k is taken in both cases to be positive. Graphs typical of $U_{\text{eff}}(r,\ell^2)$ in those two cases are shown in Figure 2. Also worthy of special mention is the "super-Coulombic case" $U(r) = -kr^{-2}$ which, though physically unimportant, acquires theoretical interest from several interrelated circumstances: the associated effective potential can be written

$$U_{\text{eff}}(r,\ell^2) = \left\{\tfrac{\ell^2}{2m} - k\right\}r^{-2}$$

which presents only a single power or r; bound orbits are possible only if $\ell^2 < 2mk$, and such orbits are unstable against collapse. Of deeper interest

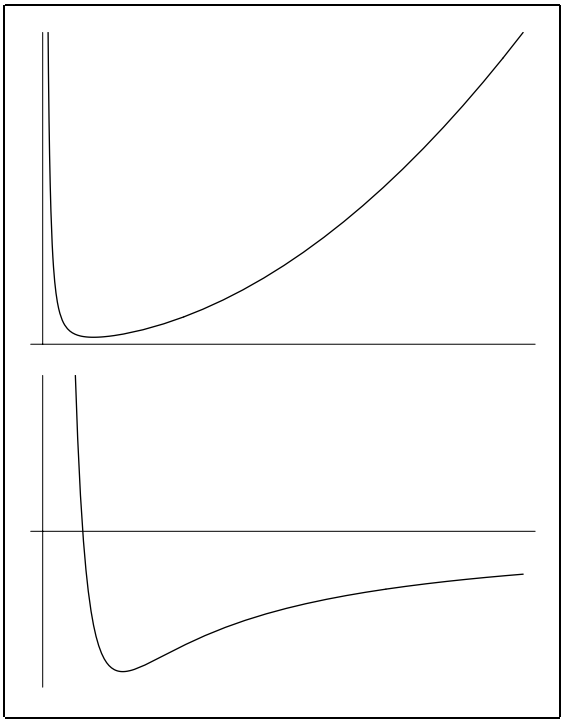


FIGURE 2: *Typical form of* $U_{\text{eff}}(r,\ell^2)$ *in the harmonic case (above) and the Coulombic case (below). In the former, all orbits are bound, with* $E>0$*. In the latter case orbits are bound or unbound according as* $E_{\odot}(\ell^2)\leqslant E<0$ *or* $0\leqslant E$*.*

is the fact that of all power-law potentials $U\sim r^n$ only the super-Coulombic potential $U\sim 1/r^2$ leads to *dilationally invariant* equations of motion,[6] but I must be content to pursue this specialized topic no further. I turn instead to discussion relating to *how it comes about* that the harmonic and Coulombic cases merit special attention.

The *circular orbit condition* can be expressed $\frac{d}{dr}U_{\text{eff}}=0$ and entails

$$r\mapsto R_{\odot}=(\ell^2/mkn)^{\frac{1}{n+2}}$$

(which, by the way, forces us to assign k such a sign as to render $kn>0$) giving

[6] For details relating to the meaning and implications of this remark, see "Dilational symmetry in non-relativistic particle mechanics" (Reed College Physics Seminar Notes, 14 November 1973), which is reproduced as an appendix to RELATIVISTIC CLASSICAL FIELDS (1973); R. Jackiw & S. Coleman, "Why dilation generators do not generate dilations," Annals of Physics **67**, 552 (1971); and R. Jackiw, "Introducing scale symmetry," Physics Today (January, 1972; see especially the illustration which appears on the cover of that issue).

$$\ell^2\text{-dependent radius of circular orbit } R_\odot = \begin{cases} (\ell^2/2mk)^{1/4} & : \text{ case } n=+2 \\ (\ell^2/mk) & : \text{ case } n=-1 \end{cases}$$

Such orbits are pursued with energy

$$E_\odot(\ell^2) = \begin{cases} \sqrt{2k\ell^2/m} \;=\; 2U(R) & : \text{ case } n=+2 \\ -\frac{1}{2}mk^2/\ell^2 = \frac{1}{2}U(R) & : \text{ case } n=-1 \end{cases}$$

and these results—written

$$E_\odot(\ell^2) = T + U(R) \quad \text{with} \quad \begin{cases} T = +1U(R) & : \text{ case } n=+2 \\ T = -\frac{1}{2}U(R) & : \text{ case } n=-1 \end{cases}$$

—are found to be consistent with assertions of the *virial theorem*.[7] Location of the turning points requires that we discover real solutions (when they exist) of

$$kr^{n+2} - Er^2 + \ell^2/2m = 0$$

which generally cannot be accomplished in closed form. But in the harmonic case we have

$$kr^4 - Er^2 + \ell^2/2m = 0$$

giving

$$r^2 = \frac{E \pm \sqrt{E^2 - 2k\ell^2/m}}{2k} \quad : \quad \text{necessarily } E^2 \geqslant 2k\ell^2/m = E_\odot^2$$

whence

$$r_{\min}\,(E,\ell^2) = \left[\frac{E - \sqrt{E^2 - 2k\ell^2/m}}{2k}\right]^{\frac{1}{2}} \leqslant R_\odot(\ell^2)$$

$$r_{\max}(E,\ell^2) = \left[\frac{E + \sqrt{E^2 - 2k\ell^2/m}}{2k}\right]^{\frac{1}{2}} \geqslant R_\odot(\ell^2)$$

which coalesce as $E \downarrow E_\odot$ and become imaginary (unphysical) if $E < E_\odot$. In the Coulombic (Keplerean) case we have

$$Er^2 + kr - \ell^2/2m = 0$$

and find that we must distinguish

- elliptic cases $E_\odot \leqslant E < 0$ from
- parabolic cases $E = 0$ from
- hyperbolic cases $E > 0$.

The relatively fussy details are developed in §6. Such details are of interest in a variety of connections, and serve indispensably as aids to setting parameters and initial conditions in numerical work such as that to which I now turn.

[7] Goldstein, §3–4. In the general case we expect, on this elegant basis, to have $T = \frac{n}{2}U(R)$.

One might naively suppose that, having selected a potential $U(r)$ and assigned values to E, ℓ^2 and $\mathring{r}(0)$, one has only to integrate (9)—numerically if not analytically—to obtain a description of the implied orbit $r(\theta)$. But a computational problem (which *Mathematica* was quick to bring to my attention) arises from the circumstance that (9) should properly be written

$$\mathring{r} = \pm\tfrac{mr^2}{\ell}\sqrt{\tfrac{2}{m}\left[E - U(r) - \tfrac{\ell^2}{2mr^2}\right]} \tag{17}$$

where the sign is $\pm$ on ascending/descending sectors of an orbit; it *flips* $+ \to -$ as $r(\theta)$ passes through $r_{\max}$, and flips again $- \to +$ as $r(\theta)$ passes through $r_{\min}$. Any successful orbit-generating algorithm based upon (17) will contain necessarily some *sign-selection sub-routine*, and this my rudimentary numerical skills have not permitted me to accomplish. I need graphical representations of orbits to make my expository point, so I temporarily retreat to the elementary Cartesian t-parameterized *dynamics* of the problem, working from

$$\left.\begin{aligned} m\ddot{x} &= -kn(x^2+y^2)^{\frac{n-2}{2}}x \\ m\ddot{y} &= -kn(x^2+y^2)^{\frac{n-2}{2}}y \end{aligned}\right\} \tag{18}$$

though it hurts to do so: (17) is a single first-order equation, phrased in terms that relate directly to the geometrical problem of interest, while (18) is a pair of second-order equations which relate only incidentally orbital geometry.

To study the qualitative implications of (18) we write those equations in simplified canonical form

$$\begin{aligned} \dot{x} &= u \\ \dot{u} &= -(x^2+y^2)^{\frac{n-2}{2}}x \\ \dot{y} &= v \\ \dot{v} &= -(x^2+y^2)^{\frac{n-2}{2}}y \end{aligned}$$

and set $x_0 = 1$, $u_0 = y_0 = 0$. If, additionally, we set $v_0 = 1$ then the resulting orbit is found in all cases—irrespective of the value assigned to n—to be a unit circle, centered at the origin. Thus nicely positioned in parameter space, we (to obtain orbits with informative shape) tweek the launch speed (set $v_0 = 0.5$) and draw the orbits which result when

- n lies in the neighborhood of its harmonic value $n = +2$;
- n lies in the neighborhood of its Coulombic value $n = -1$. The results are displayed in Figures 3 & 4, which are intended to lend experimental weight to the following claim:

Precession is the rule, orbital closure the exception. For any given power law $U \sim r^n$ the bound-orbital sector of the parameter space $\left\{E, \ell^2\right\}$ is peppered with points that give rise to closed/periodic orbits, but only in the harmonic and Coulombic cases is *every* bound orbit closed (and, as it happens, elliptical, with *center* at the force center in the former case, *focus* at the force center in

the latter case). This is the upshot of "Bertrand's theorem,"[8] which did not attract general interest until soon after the invention of quantum mechanics, when it was noticed—first by Pauli (1926)—that there appears to be a deep and significant connection between

- orbital closure,
- multiple separability of the Schrödinger equation,
- "accidental degeneracy" of the quantum mechanical energy spectrum.[9]

This circumstance accounts for the fact that it is to be best *modern* texts that one must look to find discussion of the proof of Bertrand's theorem.[10] Here we will be concerned mainly with aspects of the *multi-separability* issue.

[8] Joseph L. P. Bertrand (1822–1900), who received his doctorate at seventeen, made important contributions to pure mathematics, theoretical mechanics, thermodynamics and several other fields (he wrote on the flight of birds), was the author of several influential texts, and ultimately acquired virtually every scholarly distinction his native France had to bestow. "Bertrand's theorem" was published in Comptes Rendus **77**, 849 (1873). It was ignored by authors of most of the older monographs, but is discussed in §428 of E. J. Routh's *Treatise on the Dynamics of a Particle* (1898).

[9] For an accessible account of this pretty subject—which appears to retain much mystery—see H. V. McIntosh, "On accidental degeneracy in classical and quantum mechanics," AJP **27**, 620 (1959) and classic papers cited there; also "Symmetry and degeneracy" in *Group Theory and its Applications II*, edited by E. M. Loebl (1970). Relevant material can be found in CLASSICAL DYNAMICS, Chapter 9, pp. 61–74 (1964).

[10] Goldstein devotes his §3–6 to discussion of some implications of Bertrand's theorem, and in his Appendix A presents a detailed proof. Valuable discussion can be found also in §2.3.3 of J. V. José & E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (1998) and—which I especially recommend—§§4.4/5 of J. L. McCauley, *Classical Mechanics: Transformations, Flows, Integrable & Chaotic Dynamics* (1997). It is my impression, however, that the world still awaits the development of a truly illuminating account of the origin and ramifications of Bertrand's theorem.

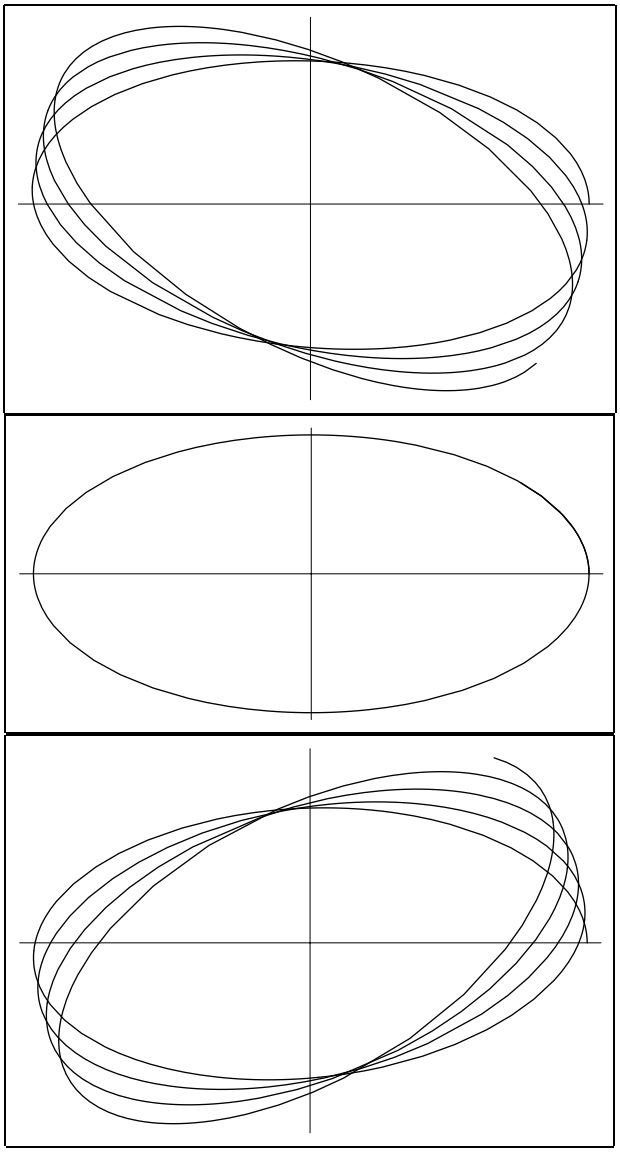
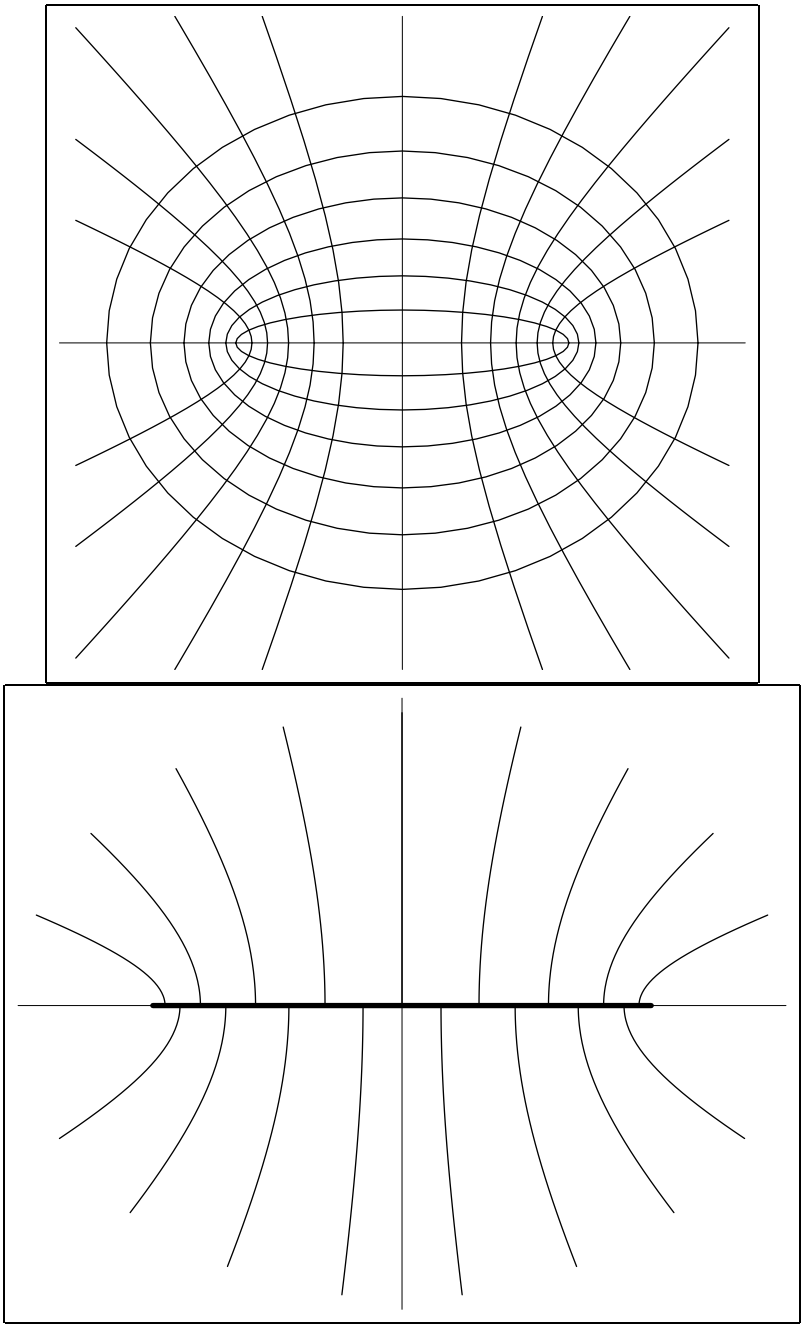


FIGURE 3: *Orbits with n lying in the neighborhood of its harmonic value $n = +2$ (central figure): precession in the top figure ($n = \frac{21}{10}$) is retrograde, precession in the bottom figure ($n = \frac{19}{20}$) is prograde.*



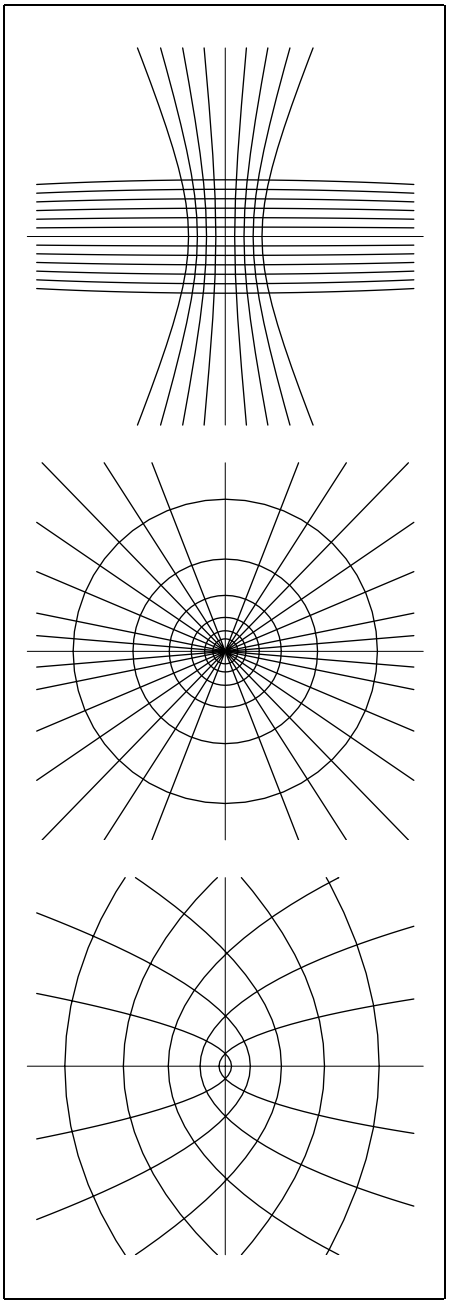


FIGURE 4: *Orbits with n lying in the neighborhood of its Coulombic value $n = -1$: precession in the top figure ($n = -\frac{95}{100}$) is retrograde, precession in the bottom figure ($n = -\frac{105}{100}$) is prograde.*

2. Reduced central force problem in "alternative polar coordinates". Later we will encounter a limiting process which leads back most naturally not to the familiar polar coordinates, as employed above, but to a slight variant of them—the so-called "alternative polar coordinate system,"[11] which arises when one sets $r = ae^s$, giving

$$\left.\begin{aligned} x &= ae^s \cos\theta \\ y &= ae^s \sin\theta \\ &\quad a \text{ is a constant of arbitrary value, with } [a] = \text{length} \end{aligned}\right\} \tag{19}$$

In this notation the Lagrangian (2.2) becomes

$$L = \tfrac{1}{2}ma^2(\dot{s}^2 + \dot{\theta}^2)e^{2s} - U(ae^s) \tag{20}$$

which acquires interest from the circumstance that it can—if we introduce

$$\begin{aligned} u_1(s) &= +ma^2e^{2s} \\ u_2(\theta) &= 0 \\ w_1(s) &= -ma^2e^{2s}U(ae^s) \\ w_2(\theta) &= 0 \end{aligned}$$

—be cast into the form

$$L = \tfrac{1}{2}\big\{u_1(s) + u_2(\theta)\big\}(\dot{s}^2 + \dot{\theta}^2) + \frac{w_1(s) + w_2(\theta)}{u_1(s) + u_2(\theta)} \tag{21}$$

characteristic of Lagrangians which are "separable in the sense of Liouville."[12] This is, as will emerge, a very great analytical advantage. It is an advantage shared by (2.1) only in the harmonic case $U(r) \sim r^2$, and shared by (2.2) in no case. I turn now to an account of how Liouville's method runs in the particular case now in hand.

We will agree at the outset to write $u = u_1 + u_2$. We multiply

$$\left\{\frac{d}{dt}\frac{\partial}{\partial\dot{s}} - \frac{\partial}{\partial s}\right\}L = \tfrac{d}{ds}(u\cdot\dot{s}) - \tfrac{1}{2}\tfrac{\partial u}{\partial s}(\dot{s}^2 + \dot{\theta}^2) - \tfrac{\partial}{\partial s}(w_1/u) = 0$$

by $2u\cdot\dot{s}$ to obtain

$$\tfrac{d}{ds}(u^2\cdot\,\dot{s}^2) - (u\cdot\dot{s})\tfrac{\partial u}{\partial s}(\dot{s}^2 + \dot{\theta}^2) - 2(u\cdot\dot{s})\tfrac{\partial}{\partial s}(w_1/u) = 0 \tag{22}$$

By energy conservation

$$\tfrac{1}{2}u\cdot(\dot{s}^2 + \dot{\theta}^2) - \frac{w_1 + w_2}{u} = E \tag{23}$$

---

[11] See P. Moon & D. E. Spencer, *Field Theory Handbook* (1964), p. 13.

[12] See §3 in Reference 4. "Alternative polar coordinates" acquire much of their interest from the circumstance that the polar Lagrangian (2.2) is *not* of Liouvillean form.

we have

$$u \cdot (\dot{s}^2 + \dot{\theta}^2) = 2\Big[E + \frac{w_1 + w_2}{u}\Big]$$

which when fed back into (22) gives

$$\begin{aligned}\tfrac{d}{dt}(u^2 \cdot \dot{s}^2) &= 2\big[E + \tfrac{w_1+w_2}{u}\big]\dot{s}\tfrac{\partial u}{\partial s} + 2(u\cdot\dot{s})\tfrac{\partial}{\partial s}(w_1/u) \\ &= 2\dot{s}\tfrac{\partial}{\partial s}\big[\big(E + \tfrac{w_1+w_2}{u}\big)u\big] \\ &= 2\dot{s}\tfrac{\partial}{\partial s}\big[E \cdot u_1 + w_1\big] \\ &= 2\tfrac{d}{dt}[E \cdot u_1(s) + w_1(s)\big]\end{aligned}$$

So we have

$$\tfrac{1}{2}(u^2 \cdot \dot{s}^2) = E \cdot u_1(s) + w_1(s) + \epsilon_1 \tag{24.1}$$

and, by an identical argument,

$$\tfrac{1}{2}(u^2 \cdot \dot{\theta}^2) = E \cdot u_2(\theta) + w_2(\theta) + \epsilon_2 \tag{24.2}$$

Here ϵ_1 and ϵ_2 are by nature constants of integration, constrained (if we are to achieve consistency with the energy relation (23)) to satisfy

$$\epsilon_1 + \epsilon_2 = 0$$

In general applications of Liouville's method equations (24) remain coupled, but very simply (through the shared u^2-factor on the left). Liouville circumvents this detail by dividing one equation into the other, obtaining

$$\frac{ds}{d\theta} = \sqrt{\frac{E \cdot u_1(s) + w_1(s) - \epsilon}{E \cdot u_2(\theta) + w_2(\theta) + \epsilon}} \tag{25}$$

All reference to t has at this point disappeared; (25) refers to the *design of the trajectory*, but information pertaining to temporal progress along that trajectory can be recovered from energy conservation. In the particular case at hand several special circumstances (the θ-independence of $u(r,\theta)$; the fact that u_2 and w_2 both vanish) make it possible to argue somewhat more sharply: we have

$$\begin{aligned}\tfrac{1}{2}(ma^2e^{2s})^2 \cdot \dot{s}^2 &= E \cdot (ma^2e^{2s}) - ma^2e^{2s}U(ae^s) - \epsilon \\ \tfrac{1}{2}(ma^2e^{2s})^2 \cdot \dot{\theta}^2 &= \quad 0 \quad + \quad 0 \quad + \epsilon\end{aligned}$$

From the latter equation, notated $\frac{1}{2}(mr^2\dot{\theta})^2 = \epsilon$, we learn that

$$\epsilon = \tfrac{1}{2}\ell^2 \tag{26}$$

The former equation can therefore be notated

$$\tfrac{1}{2}m^2r^2\dot{r}^2 = Emr^2 - mr^2U(r) - \tfrac{1}{2}\ell^2$$

which upon division by $\frac{1}{2}m^2r^2$ gives back (8). Finally, (25) becomes

$$\begin{aligned}\frac{ds}{d\theta} = \frac{1}{r}\frac{dr}{d\theta} &= \sqrt{\frac{E \cdot ma^2e^{2s} - ma^2e^{2s}U(ae^s) - \frac{1}{2}\ell^2}{\frac{1}{2}\ell^2}} \\ &= \tfrac{mr}{\ell}\sqrt{\tfrac{2}{m}\Big[E - U(r) - \tfrac{\ell^2}{2mr^2}\Big]}\end{aligned}$$

which reproduces the orbital equation (9).

3. Kepler problem in confocal parabolic coordinates. The coordinate system in question arises when one writes[13]

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\mu^2-\nu^2) \\ y &= \mu\nu \end{aligned}\right\} \quad \text{which entail} \quad r = \tfrac{1}{2}(\mu^2+\nu^2) \tag{27}$$

The reduced Lagrangian (2.1) becomes

$$L = \tfrac{1}{2}m(\mu^2+\nu^2)(\dot\mu^2+\dot\nu^2) - U(r)$$

which—and this is the point—assumes Liouville's design in the Coulombic case: we have

$$\downarrow$$

$$L = \tfrac{1}{2}m(\mu^2+\nu^2)(\dot\mu^2+\dot\nu^2) + \frac{2k}{\mu^2+\nu^2} \tag{28}$$

which is of the form (21) with

$$\left.\begin{aligned} u_1(\mu) &= m\mu^2 \\ u_2(\nu) &= m\nu^2 \\ w_1(\mu) &= mk-\kappa \\ w_2(\nu) &= mk+\kappa \end{aligned}\right\} \tag{29}$$

where κ is a dimensioned constant of arbitrary value, destined immediately to be absorbed into the definition of a separation constant. For Liouville's argument culminates in an orbital equation (see again (25)) which in the present instance reads

$$\begin{aligned} \frac{d\mu}{d\nu} &= \sqrt{\frac{E\cdot u_1(\mu)+w_1(\mu)-\epsilon}{E\cdot u_2(\nu)+w_2(\nu)+\epsilon}} \\ &= \sqrt{\frac{E\cdot m\mu^2+mk-\kappa-\epsilon}{E\cdot m\nu^2+mk+\kappa+\epsilon}} \\ &= \sqrt{\frac{E\cdot m\mu^2+mk-\varepsilon}{E\cdot m\nu^2+mk+\varepsilon}} \quad \text{with} \quad \varepsilon \equiv \epsilon+\kappa \end{aligned} \tag{30}$$

I turn now to remarks intended to expose the answer to this question: What is the physical significance—what, more properly (since it arises here from a statement referring to the *geometry* of orbits), is the geometrical significance—of the separation constant ε?[14]

The equation $y^2 = 2p\cdot(x-a)$ describes a parabola which opens to the right or left according as $p \gtrless 0$, has focus placed at $x_{\text{focus}} = a + \frac{1}{2}p$, and intercepts the x-axis at $x_{\text{intercept}} = a$. From (27) we obtain

$$\begin{aligned} y^2 &= \mu^2(\mu^2-2x) \\ y^2 &= \nu^2(\nu^2+2x) \end{aligned}$$

[13] See Moon & Spencer, Reference 11, p. 21.

[14] Note the typographic distinction between `epsilon` ϵ and `varepsilon` ε.

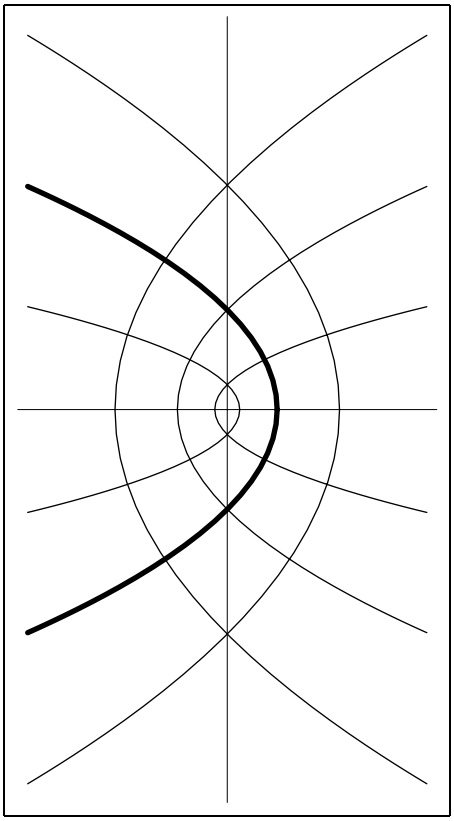


FIGURE 5: *Confocal parabolic coordinate system. Curves of constant* μ *open to the left and cross the* x*-axis at*

$$x_{\text{intercept}} = +\tfrac{1}{2}\mu^2$$

while curves of constant ν *open to the right and cross the* x*-axis at*

$$x_{\text{intercept}} = -\tfrac{1}{2}\nu^2$$

Each $\supset$*-parabola intersects each* $\subset$*-parabola twice, at intersection points which the sign of* ν *serves to distinguish; by that convention points of the* $\{x, y\}$*-plane come into bi-unique correspondence with points on the right half of the* $\{\mu, \nu\}$*-plane, and the natural range of the parabolic coordinates becomes* $0 \leqslant \mu < \infty$, $-\infty < \nu < +\infty$.

of which the former describes a μ-parameterized family of confocal parabolas (each has its focus situated at the origin) opening to the left, and the latter a ν-parameterized family of confocal parabolas opening to the right, as shown above. We are returned to the physics of the matter when we ask: Under what conditions can a coordinate parabola—taken, let us say, to be a μ-parabola (such as the one emphasized in the figure)—be identified with a Keplerean orbit?

4. The confocal elliptic coordinate system. The coordinate system now in question arises when one writes[15]

$$\left.\begin{aligned} x &= a\cosh\xi\cos\eta \\ y &= a\sinh\xi\sin\eta \end{aligned}\right\} \tag{31}$$

and assumes $\{\xi,\eta\}$ to range on a semi-infinite strip: $0 \leqslant \xi < \infty$, $0 \leqslant \eta < 2\pi$. Elimination first of η, then of ξ, gives

$$\left.\begin{aligned} \left(\frac{x}{a\cosh\xi}\right)^2 + \left(\frac{y}{a\sinh\xi}\right)^2 &= 1 \\ \left(\frac{x}{a\cos\eta}\right)^2 - \left(\frac{y}{a\sin\eta}\right)^2 &= 1 \end{aligned}\right\} \tag{32}$$

according to which curves of constant ξ are ellipses, with foci at $x = \pm a$ and x-intercepts at $x = \pm a\cosh\xi$, y-intercepts at $y = \pm a\sinh\xi$. Curves of constant η are confocal hyperbolæ, with x-intercepts at $x = \pm a\cos\xi$ (the y-intercepts are imaginary). The elliptic coordinate system—which might more properly be (and occasionally is) called the "confocal conic coordinate system"—is shown in Figure 6. That figure was obtained from (32) with the aid of *Mathematica*'s `ImplicitPlot` resource, and in one important respect misrepresents the situation, which can be expressed variously as follows: to assign η a fixed value $\eta \in [0, \frac{\pi}{2}]$ and then to allow ξ to range $0 \to \xi \to \infty$ is to obtain only the "upper right quadrant" of a coordinate hyperbola (see Figure 7); to obtain the lower quadrant of that branch send $\eta \mapsto 2\pi - \eta$; to obtain the upper (else lower) branch of that hyperbola send $\eta \mapsto \pi - \eta$ (else $\eta \mapsto (2\pi - (\pi - \eta)) = \pi + \eta$); the η-address of a hyperbola *exhibits a jump-discontinuity* as one passes from the upper to the lower quadrants of that branch—at the point, that is to say, where the hyperbolic branch intersects the line joining the foci.

The elliptic coordinate system gives back Cartesian, (alternate) polar and parabolic coordinates as limiting cases. The details are most easily discussed with the aid of methods borrowed from the theory of functions of a complex variable. We proceed from the observation that (31) entails

$$z = x + iy = a\cosh\omega$$
$$\omega \equiv \xi + i\eta$$

The function $\cosh\omega$ is analytic, so the map

$$\omega \mapsto z \quad : \quad \text{semi-infinite strip on } \omega\text{-plane} \mapsto \text{entire } z\text{-plane}$$

is conformal; this accounts for the fact that the curvilinear gridlines shown in Figure 6 intersect orthogonally. Small patches of the elliptic coordinate system therefore look Cartesian; at the top of Figure 8 I show the Cartesian neighborhood of the z-origin. The figure was produced by placing the foci very far away, i.e., by setting $a \gg$ figure-dimension.

[15] See Moon & Spencer, Reference 17, p. 21.

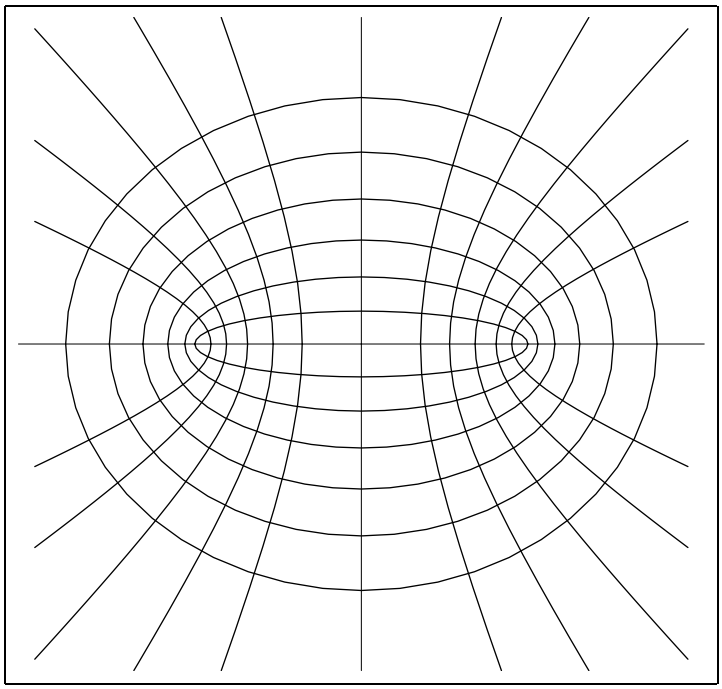


FIGURE 6: *Elliptic (or confocal conic) coordinate system. ξ is constant on each ellipse; η is constant on each hyperbolic quadrant. The location of the foci is apparent. The figure was generated by (32) with the aid of Mathematica's* `ImplicitPlot` *package.*

The elliptic system becomes alternate polar when "seen from very far away," i.e., when the foci are placed very close together ($a \ll$ figure-dimension), as they have been in the central image of Figure 8. To establish the point analytically we have simply to observe that

$$\begin{aligned} z &= a\cosh\omega \\ &\sim \tfrac{1}{2}ae^{\xi}\cdot(\cos\eta + i\sin\eta) \quad \text{in the approximation that } e^{-\xi}\sim 0 \end{aligned}$$

at which point we have in effect recovered (19). Parabolic coordinates are recovered when the elliptic system is examined in the close vicinity of either focus. For if we introduce $Z = z - a$ (origin now placed at the right focus) and work in the approximation that $|Z| \ll a$ we have

$$Z = a(1 + \tfrac{1}{2}\omega^2 + \cdots) - a \sim \tfrac{1}{2}\Omega^2$$
$$\Omega \equiv \sqrt{a}\omega \equiv \mu + i\nu$$

giving

$$Z \equiv X + iY = \tfrac{1}{2}(\mu^2 - \nu^2) + i\mu\nu$$

at which point we have recovered precisely (27).

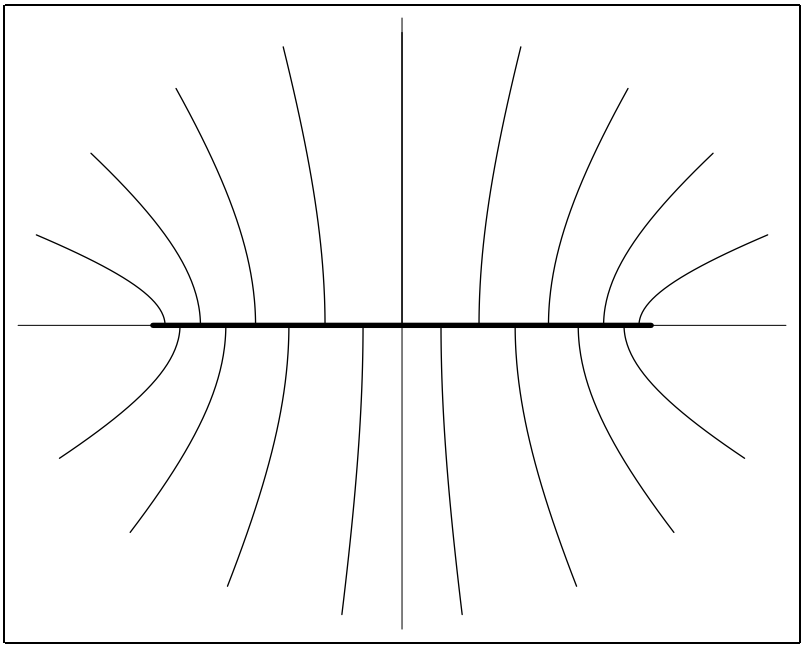


FIGURE 7: *The coordinate η serves to identify only* one quadrant *of a hyperbola. An entire hyperbola (both branches) is obtained when one specifies the value not of η but of* $\cos\eta$. *The heavy line—the "line of η-discontinuity"—joins the foci. The figure was generated by (31) with the aid of Mathematica's* `ParametricPlot` *resource.*

In the intended physical application we will want find it natural to place the force center at one focus—let us again say the right focus—and to let the other focus "float," writing $\boldsymbol{A}$ to describe the position of the "empty" focus relative to a Cartesian frame erected at the physical focus. $\boldsymbol{A}$ is a vector of length $2a$. We are led thus to an $\boldsymbol{A}$-parameterized *population* of elliptic coordinate systems, each of which shares one focus, and of which a typical member is illustrated in Figure 9. To describe analytically the modified elliptic coordinate system in question we have only to write

$$z = e^{i\alpha} \cdot a(\cosh\omega - 1) \tag{33}$$

Then

$$\omega = 0 \mapsto z = 0 \quad : \quad \text{describes placement of the force center}$$

while

$$\begin{aligned} \omega = i\pi \mapsto z &= -2ae^{i\alpha} \\ &= 2ae^{i(\alpha+\pi)} \quad : \quad \text{describes placement } \boldsymbol{A} \text{ of the "empty focus"} \end{aligned}$$

The parameter α refers to a degree of coordinate freedom which it is important to bear in mind, though in practical work it often proves expedient to set $\alpha = 0$.

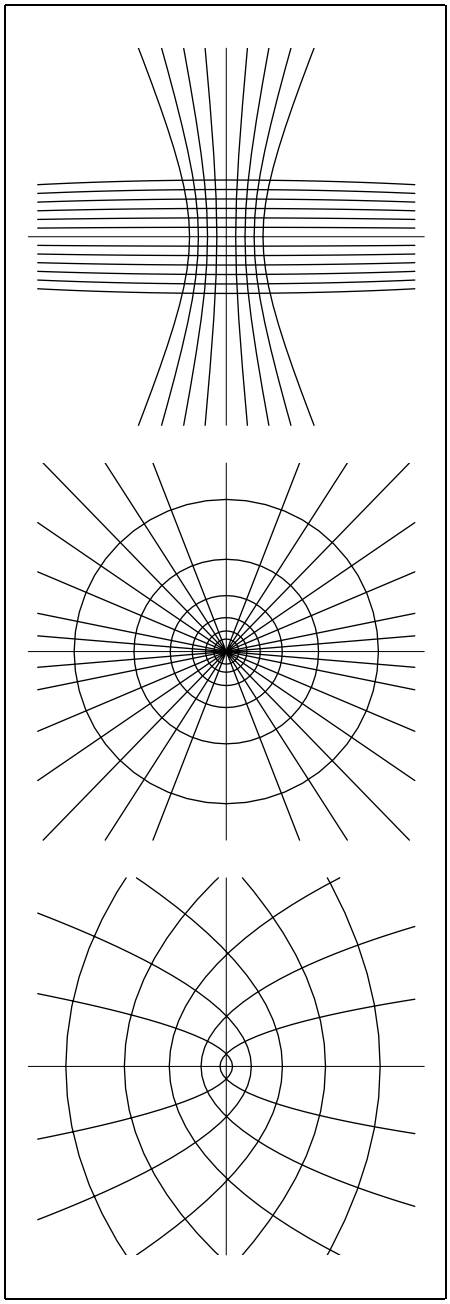


FIGURE 8: *Cartesian, alternate polar and parabolic limits of the elliptic coordinate system, got by magnifying the region near the origin (top), by placing the foci very close together (middle), by magnifying the region near the left focus (bottom).*

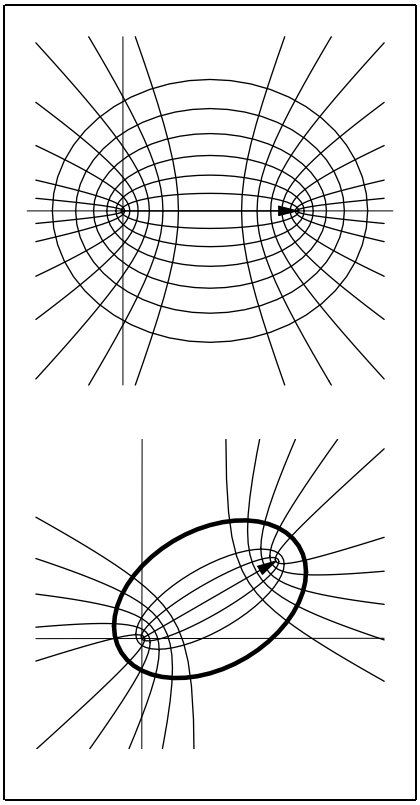


FIGURE 9: *Keplerean modification of the elliptic coordinate system. The standard system has been first translated (top figure) so as to place a focus at the origin (force center), and then rotated (bottom figure). The heavy ellipse alludes to the possibility of identifying coordinate lines with Keplerean orbits.*

The polar and parabolic coordinate systems give rise to populations of coordinate lines which can in both cases be associated with subsets of the set of all possible Keplerean orbits (i.e., with the set of all conic sections which have a focus at the origin). It is, in this respect, a striking—and potentially useful—property of the elliptic system(s) that

> *every* Keplerean orbit can be associated with one or another of the coordinate lines supplied by some elliptic system;

one has only to assign appropriate values to $\boldsymbol{A}$ and ξ (else η). Moreover, the vector $\boldsymbol{A}$ (which is to say: the location of the empty focus) acquires the status of a *constant of the motion*. I return to this topic in §6.

With all preparations now behind us, we are in position at last to turn to review of the formal mechanical essentials of the topic announced in the title:

5. Physical application: the Kepler problem. The reduced Lagrangian (1) can, in the Keplerean case,[16] be notated

$$\begin{aligned} L &= \tfrac{1}{2}m\dot{z}^*\dot{z} + k\frac{1}{\sqrt{z^*z}} \\ &= \tfrac{1}{2}ma^2(\sinh\omega)^*(\sinh\omega)\cdot\dot{\omega}^*\dot{\omega} + k\frac{1}{a\sqrt{(\cosh\omega-1)^*(\cosh\omega-1)}} \end{aligned}$$

which with the aid of

$$\begin{aligned} \sinh\omega &= \sinh\xi\cos\eta + i\cosh\xi\sin\eta \\ \cosh\omega &= \cosh\xi\cos\eta + i\sinh\xi\sin\eta \\ &\Downarrow \\ (\sinh\omega)^*(\sinh\omega) &= \cosh^2\xi - \cos^2\eta \\ (\cosh\omega-1)^*(\cosh\omega-1) &= (\cosh\xi-\cos\eta)^2 \end{aligned}$$

becomes

$$L = \tfrac{1}{2}ma^2(\cosh^2\xi-\cos^2\eta)(\dot{\xi}^2+\dot{\eta}^2) + k\frac{\cosh\xi+\cos\eta}{a(\cosh^2\xi-\cos^2\eta)} \tag{34}$$

But (34) is of the design (21) with

$$\left.\begin{aligned} u_1(\xi) &= +ma^2\cosh^2\xi \\ u_2(\eta) &= -ma^2\cos^2\eta \\ w_1(\xi) &= kma\ \cosh\xi \\ w_2(\xi) &= kma\ \cos\eta \end{aligned}\right\} \tag{35}$$

and is therefore "separable in the sense of Liouville." The remarkable implication is that the Kepler problem is separable not only in polar and parabolic coordinates (separability in those cases is well known) but in each of the confocal elliptic systems (33), where "each" means "irrespective of the values ascribed to a and to α" (in short: for all $\boldsymbol{A}$). Polar/parabolic separability can be understood to arise as limiting consequences of this exceptional fact.

Working from (33) we find that the momenta conjugate to ξ and η can be described

$$\begin{aligned} p_\xi &= u\dot{\xi} \\ p_\eta &= u\dot{\eta} \\ u &\equiv u_1(\xi)+u_2(\eta) = ma^2(\cosh^2\xi-\cos^2\eta) \end{aligned}$$

[16] Generally I reserve "Keplerean" for problems in which the interaction is (attractive) *gravitational* ($k=GMm$), and—as previously—use "Coulombic" when the sign and physical interpretation of k are non-specific. In the present context I find it artificial to maintain that convention.

and that the associated Hamiltonian $H = p_\xi\dot{\xi} + p_\eta\dot{\eta} - L$ can therefore be rendered

$$\begin{aligned} H &= \frac{1}{2u}\Big\{\big(p_\xi^2 + p_\eta^2\big) - 2(w_1 + w_2)\Big\} \\ &= \frac{1}{2ma^2}\left\{\frac{1}{\cosh^2\xi - \cos^2\eta}(p_\xi^2 + p_\eta^2) - 2kma\frac{1}{\cosh\xi - \cos\eta}\right\} \end{aligned} \tag{36}$$

Eliminating E between Liouville's equations (24)—which in the present instance read

$$\begin{aligned} \tfrac{1}{2}(u^2\cdot\dot{\xi}^2) &= E\cdot u_1(\xi) + w_1(\xi) - \epsilon \\ \tfrac{1}{2}(u^2\cdot\dot{\eta}^2) &= E\cdot u_2(\eta) + w_2(\eta) + \epsilon \end{aligned}$$

—we obtain

$$\begin{aligned} \epsilon &= -\tfrac{1}{2}u\cdot(u_2\dot{\xi}^2 - u_1\dot{\eta}^2) + \frac{u_2w_1 - u_1w_2}{u} \\ &\downarrow \\ G &= -\frac{1}{2u}\Big\{\big(u_2p_\xi^2 - u_1p_\eta^2\big) - 2\big(u_2w_1 - u_1w_2\big)\Big\} \\ &= \tfrac{1}{2}\left\{\frac{1}{\cosh^2\xi - \cos^2\eta}\big(p_\xi^2\cos^2\eta + p_\eta^2\cosh^2\xi\big) - 2kma\,\frac{\cosh\xi\cos\eta}{\cosh\xi - \cos\eta}\right\} \end{aligned} \tag{37}$$

Here ϵ, which came to us as a "separation constant in the sense of Liouville," has been promoted to the status of an observable, which I will call "Liouville's observable." With the indispensable assistance of *Mathmatica* we confirm—whether we work from the generic or the elliptic-specific versions of (36) and (37)—that the Poisson bracket

$$[H, G\,] = 0$$

according to which ϵ acquires this interpretation:

$$\epsilon = \text{dynamically conserved value of } G(p_\xi, p_\eta, \xi, \eta)$$

But that observation, while it shifts the locus, leaves unanswered the question: What is the "meaning" of ϵ? I approach the question by specialization of the methods and results developed in §§4–6 of an essay previously cited.[1]

Keplerean orbits are standardly classified by specification of the conserved values of H, $\boldsymbol{L}$ and $\boldsymbol{K}$, where

$$\begin{aligned} H &\equiv \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} - \tfrac{k}{r} && : \text{ Hamiltonian} \\ \boldsymbol{L} &\equiv \boldsymbol{r}\times\boldsymbol{p} && : \text{ angular momentum vector} \\ \boldsymbol{K} &\equiv \tfrac{1}{m}(\boldsymbol{p}\times\boldsymbol{L}) - \tfrac{k}{r}\boldsymbol{r} \\ &= \tfrac{1}{m}\big[(\boldsymbol{p}\cdot\boldsymbol{p})\boldsymbol{r} - (\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}\big] - \tfrac{k}{r}\boldsymbol{r} && : \text{ Lenz vector} \end{aligned} \tag{38}$$

and only the last of those is at all unfamiliar.[17] Clearly $\boldsymbol{K} \perp \boldsymbol{L}$, and since $\boldsymbol{L}$ stands normal to the orbital plane, $\boldsymbol{K}$ lies *in* the orbital plane. We have, by convention, identified the orbital plane with the $\{x, y\}$-plane, and therefore have

$$\boldsymbol{L} = \begin{pmatrix} 0 \\ 0 \\ L_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ xp_y - yp_x \end{pmatrix}$$

and

$$\boldsymbol{K} = \begin{pmatrix} K_x \\ K_y \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{m} p_y (xp_y - yp_x) - \frac{k}{r} x \\ \frac{1}{m} p_x (yp_x - xp_y) - \frac{k}{r} y \\ 0 \end{pmatrix}$$

If in (33) we set $\alpha = 0$ we have

$$\begin{aligned} x &= a \cosh\xi \cos\eta - a \\ y &= a \sinh\xi \sin\eta \end{aligned}$$

and compute

$$\begin{aligned} p_\xi &= \tfrac{\partial x}{\partial \xi} p_x + \tfrac{\partial y}{\partial \xi} p_y \\ &= a\big\{ + p_x \sinh\xi \cos\eta + p_y \cosh\xi \sin\eta \big\} \\ p_\eta &= \tfrac{\partial x}{\partial \eta} p_x + \tfrac{\partial y}{\partial \eta} p_y \\ &= a\big\{ - p_x \cosh\xi \sin\eta + p_y \sinh\xi \cos\eta \big\} \end{aligned}$$

which by matrix inversion yields

$$\begin{aligned} p_x &= \tfrac{1}{a(\cosh^2\xi - \cos^2\eta)} \big\{ p_\xi \sinh\xi \cos\eta - p_\eta \cosh\xi \sin\eta \big\} \\ p_y &= \tfrac{1}{a(\cosh^2\xi - \cos^2\eta)} \big\{ p_\xi \cosh\xi \sin\eta + p_\eta \sinh\xi \cos\eta \big\} \end{aligned}$$

This information puts us in position to compute (for example, and because it will soon prove useful)

$$L_z = -p_\xi \tfrac{\sin\eta}{\cosh\xi + \cos\eta} + p_\eta \tfrac{\sinh\xi}{\cosh\xi + \cos\eta} \tag{39}$$

With foreknowledge of where I'm headed, I compute

$$ma^2 H = \tfrac{1}{2(\cosh^2\xi - \cos^2\eta)} (p_\xi^2 + p_\eta^2) - kma \tfrac{1}{\cosh\xi - \cos\eta} \tag{40}$$

$$\begin{aligned} maK_x = &- \tfrac{1}{\cosh^2\xi - \cos^2\eta} \Big\{ p_\xi \cosh\xi \sin\eta + p_\eta \sinh\xi \cos\eta \Big\} \\ &\cdot \Big\{ p_\xi \tfrac{\sin\eta}{\cosh\xi + \cos\eta} - p_\eta \tfrac{\sinh\xi}{\cosh\xi + \cos\eta} \Big\} - kma \tfrac{\cosh\xi\cos\eta - 1}{\cosh\xi - \cos\eta} \end{aligned} \tag{41}$$

[17] Goldstein, in his excellent §3–9, reports that the classical physics of $\boldsymbol{K}$ was known already to Laplace in 1799, and rediscovered by Hamilton in 1845. Runge's contribution (1919) was merely expository, but was cited by Lenz in the first quantum mechanical application (1924) of Laplace's idea.

and notice that $G - ma^2H - maK_x$ is k-independent; in fact

$$\begin{aligned} G - ma^2H - maK_x &= \tfrac{1}{2(\cosh^2\xi - \cos^2\eta)}\Big\{(p_\xi^2\cos^2\eta + p_\eta^2\cosh^2\xi) \\ &\qquad - (p_\xi^2 + p_\eta^2) + 2\big[p_\xi\cosh\xi\sin\eta + p_\eta\sinh\xi\cos\eta\big] \\ &\qquad\qquad \cdot \big[p_\xi\tfrac{\sin\eta}{\cosh\xi+\cos\eta} - p_\eta\tfrac{\sinh\xi}{\cosh\xi+\cos\eta}\big]\Big\} \\ &= \tfrac{1}{2}\big[p_\xi\tfrac{\sin\eta}{\cosh\xi+\cos\eta} - p_\eta\tfrac{\sinh\xi}{\cosh\xi+\cos\eta}\big]^2 \\ &= \tfrac{1}{2}L_z^2 \end{aligned}$$

The pretty implication is that

$$G = ma^2H + maK_x + \tfrac{1}{2}L_z^2 \tag{42}$$

which in Cartesian coordinates reads

$$\begin{aligned} G = ma^2\Big\{\tfrac{1}{2m}(p_x^2 + p_y^2) - k\tfrac{1}{\sqrt{x^2+y^2}}\Big\} \\ + ma\Big\{\tfrac{1}{m}p_y(xp_y - yp_x) - k\tfrac{1}{\sqrt{x^2+y^2}}\,x\Big\} + \tfrac{1}{2}(xp_y - yp_x)^2 \end{aligned} \tag{43}$$

The unaccompanied entry of K_x into (42)—what happened to K_y?—is accounted for by the circumstance that when we set $\alpha = 0$ we identified the x-axis with the focal axis (the line joining the foci). If we introduce

$$\boldsymbol{a} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = -\tfrac{1}{2}\boldsymbol{A}$$

$=$ displacement vector: **center** of confocal conics $\longrightarrow$ **occupied focus**

then (42) becomes

$$G = m(\boldsymbol{a}\cdot\boldsymbol{a})H + m(\boldsymbol{a}\cdot\boldsymbol{K}) + \tfrac{1}{2}(\boldsymbol{L}\cdot\boldsymbol{L}) \tag{44}$$

which is manifestly invariant with respect to occupied-focus-preserving rotations of the orbital plane into itself.[18]

The argument which led to the construction (44) carries through in each of a *continuum* of confocal conic coordinatizations of the orbital plane; Liouville's observable G describes a constant of Keplerean motion *irrespective of the value ascribed to* $\boldsymbol{a}$:

$[G, H] = 0$: all $\boldsymbol{a}$

$\Downarrow$

$[H, H] = 0$: trivial, though Liouville made explicit *use* of E-conservation

$[\boldsymbol{K}, H] = \boldsymbol{0}$: might serve to *motivate the definition* of K_x, K_y

$[L_z, H] = 0$: might serve to *motivate the definition* of L_z

[18] By natural extension $O(2) \to O(3)$ we can, in fact, drop the final phase.

I am aware of no other context in which one obtains "several conservation laws for the price of one" in quite this way (i.e., as successive coefficients of a "conserved polynomial"), though it is commonplace to obtain multiple conservation laws from a single multiply-parameterized symmetry group. Thus, for example, do the three conserved components of $\boldsymbol{L}$ arise from symmetry with respect to the proper rotation group $O(3)$ of coordinate transformations. We remind ourselves of one familiar implication of this latter fact: elements of $O(3)$ can be described

$$\mathbb{R} = \exp\begin{pmatrix} 0 & -\lambda_3 & +\lambda_2 \\ +\lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & +\lambda_1 & 0 \end{pmatrix} = \exp\left\{\lambda_1\mathbb{A}_1 + \lambda_2\mathbb{A}_2 + \lambda_3\mathbb{A}_3\right\}$$

and by computation

$$[\mathbb{A}_1, \mathbb{A}_2] = \mathbb{A}_3, \quad [\mathbb{A}_2, \mathbb{A}_3] = \mathbb{A}_1 \quad \text{and} \quad [\mathbb{A}_3, \mathbb{A}_1] = \mathbb{A}_2$$

On the other hand we have the Poisson bracket relations

$$[L_x, L_y] = L_z, \quad [L_y, L_z] = L_x \quad \text{and} \quad [L_z, L_x] = L_y \tag{45}$$

which in an obvious sense "echo the design" of the underlying symmetry group. It is to gain insight into the transformation-theoretic origin of $\left\{K_x, K_y, L_z\right\}$ that we now play the game in reverse, computing

$$[K_x, K_y] = -\tfrac{2}{m}H \cdot L_z, \quad [K_y, L_z] = K_x \quad \text{and} \quad [L_z, K_x] = K_y \tag{46}$$

The equation $H(x, y, p_x, p_y) = E$ partitions 4-dimensional phase space into disjoint 3-dimensional surfaces Σ_E. The observables K_x, K_y and L_z can be interpreted to be the Lie-generators of canonical transformations which (since each commutes with H) send each such Σ_E onto itself. The orbits inscribed on Σ_E are hyperbolic/parabolic/elliptic according as E is greater than, equal to or less than zero. Let observables J_x and J_y be defined

$$J_x \equiv \begin{cases} K_x \Big/ \sqrt{+\tfrac{2}{m}H} & \text{on hyperbolic sector of phase space} \\ K_x \Big/ \sqrt{-\tfrac{2}{m}H} & \text{on elliptic sector of phase space} \end{cases}$$

$$J_y \equiv \begin{cases} K_y \Big/ \sqrt{+\tfrac{2}{m}H} & \text{on hyperbolic sector of phase space} \\ K_y \Big/ \sqrt{-\tfrac{2}{m}H} & \text{on elliptic sector of phase space} \end{cases}$$

The Poisson bracket relations (46) can then be written

$$[J_x, J_y] = -L_z, \quad [J_y, L_z] = J_x, \quad [L_z, J_x] = J_y \quad \text{on hyperbolic sector} \tag{47.1}$$

$$[J_x, J_y] = +L_z, \quad [J_y, L_z] = J_x, \quad [L_z, J_x] = J_y \quad \text{on elliptic sector} \tag{47.2}$$

From (47.2) we infer that $\{J_x, J_y, L_z\}$ generate within each elliptic Σ_E a canonical representation of $O(3)$. What of (47.1)? Proper 3×3 Lorentz matrices can be described

$$\mathbb{L} = \exp\begin{pmatrix} 0 & +\lambda_3 & +\lambda_2 \\ +\lambda_3 & 0 & -\lambda_1 \\ +\lambda_2 & +\lambda_1 & 0 \end{pmatrix} = \exp\left\{\lambda_1\mathbb{B}_1 + \lambda_2\mathbb{B}_2 + \lambda_3\mathbb{B}_3\right\}$$

and by computation we have

$$[\,\mathbb{B}_1, \mathbb{B}_2] = -\mathbb{B}_3, \quad [\,\mathbb{B}_2, \mathbb{B}_3] = \mathbb{B}_1 \quad \text{and} \quad [\,\mathbb{B}_3, \mathbb{B}_1] = \mathbb{B}_2$$

from which we infer that $\{J_x, J_y, L_z\}$ generate within each hyperbolic Σ_E a canonical representation of the *Lorentz* group $O(1,2)$. It will be appreciated that the occurance of $O(1,2)$ in such a context has no more to do with relativity than the occurance of $O(3) \equiv O(3,0)$ has to do with spatial rotation: these are simply continuous groups of low order which have found here some additional work to do—work situated not in spacetime but in phase space.

In the parabolic sector (46) reads

$$[K_x, K_y] = 0, \quad [K_y, L_z] = K_x \quad \text{and} \quad [L_z, K_x] = K_y$$

and the introduction of $\{J_x, J_y\}$ becomes unfeasible. Description of the group which $\{K_x, K_y, L_z\}$ serve to generate within Σ_0 requires special discussion, to which I may return on another occasion.

Consider again the concatenated circumstances that brought us to this point:

- we elected to work in elliptic coordinates;
- we found we were in position to exploit Liouville's method;
- we promoted Liouville's separation constant to the status of an observable;
- we were able to obtain "several conservation laws for the price of one"

because in the Keplerean application one focus remained free-floating. In contexts (Euler's "problem of two centers") where the physics serves to pin *both* foci we loose access to the line of argument which served to bring (not only L_z but also) K_x (and by implication K_y) spontaneously to our attention. And in the present (Keplerean) context we are—once L_z, K_x and K_y *have* been delivered into our hands—free to abandon the coordinates which did the deed. What becomes of those conservation laws—what survives of the argument which gave them—if we elect to work in polar (or parabolic) coordinates? The question acquires practical interest from the circumstance that those are the coordinate systems most commonly encountered. And it acquires formal interest from the circumstance that the latter coordinate systems retain only pale vestiges of the "floating focus." Looking first to details associated with the adoption of *alternate polar* coordinates:

I begin by assembling and enlarging upon some facts already in our possession: we had

$$x = ae^s \cos\theta$$
$$y = ae^s \sin\theta$$

at (19), and working from the Keplerean instance of (20) obtain

$$\begin{aligned} p_s &= ma^2\dot{s}e^{2s} \\ p_\theta &= ma^2\dot{\theta}e^{2s} \\ p_x &= (p_s\cos\theta - p_\theta\sin\theta)/ae^s \\ p_y &= (p_s\sin\theta + p_\theta\cos\theta)/ae^s \\ u_1(s) &= ma^2e^{2s} \\ u_2(\theta) &= 0 \\ w_1(s) &= kmae^s \\ w_2(\theta) &= 0 \end{aligned}$$

Working most efficiently from the generic description (37) of G we have

$$\begin{aligned} G_{\text{alternate polar}} &= -\tfrac{1}{2u}\Big\{\big(u_2p_s^2 - u_1p_\theta^2\big) - 2\big(u_2w_1 - u_1w_2\big)\Big\} \\ &= \tfrac{1}{2}p_\theta^2 \end{aligned}$$

while the generic description (36) of H and the Cartesian descriptions of L_z, K_x and K_y give

$$\begin{aligned} H &= \tfrac{1}{2u}\Big\{\big(p_s^2 + p_\theta^2\big) - 2\big(w_1 + w_2\big)\Big\} \\ &= \tfrac{1}{ma^2}\Big\{\tfrac{1}{2}e^{-2s}\big(p_s^2 + p_\theta^2\big) - kmae^{-s}\Big\} \qquad (48.1) \\ L_z &= xp_y - yp_x \\ &= p_\theta \qquad (48.2) \\ K_x &= \tfrac{1}{m}p_y(xp_y - yp_x) - \tfrac{k}{r}x \\ &= \tfrac{1}{ma}\Big\{e^{-s}\big(p_\theta^2\cos\theta + p_sp_\theta\sin\theta\big) - kma\cos\theta\Big\} \qquad (48.3) \\ K_y &= \tfrac{1}{m}p_x(yp_x - xp_y) - \tfrac{k}{r}y \\ &= \tfrac{1}{ma}\Big\{e^{-s}\big(p_\theta^2\sin\theta - p_sp_\theta\cos\theta\big) - kma\sin\theta\Big\} \qquad (48.4) \end{aligned}$$

which bring me to the delicate point of this discussion. Working from (42) one has

$$\begin{aligned} \lim_{a\downarrow 0} G_{\text{elliptic}} &= \tfrac{1}{2}L_z^2 \\ &= G_{\text{alternate polar}} \qquad (49) \end{aligned}$$

but if one introduces (48) into (42) one obtains an expression of the form

$$G_{\text{elliptic}} = a\text{-independent function of } \big\{s, \theta, p_s, p_\theta\big\} + a\cdot(\text{dangling term})$$

which does *not* give back (49) in the limit $a \downarrow 0$. Why? Because some a-factors are sequestered—folded into the *definitions* of $\big\{s, \theta, p_s, p_\theta\big\}$. When we

elect to adopt alternate polar coordinates we are led not to (42) but to the equation just prior to (48), as was anticipated already at (28); $\boldsymbol{K}$-conservation is not brought spontaneously to our attention, and we have in fact no reason to develop interest in G_{elliptic}, which simply does not appear on our radar screen. Having elected to work with a limiting case of the elliptic system we have sacrificed the analytical leverage which derives from the "floating focus; " all alternate polar coordinate systems are unipolar, and the presence of the freely-adjustable constant a appears to purchase no analytical advantage.

Adoption of *ordinary* polar coordinates places one at an even greater disadvantage. Formulæ analogous to (49) are easily developed,[19] but (as was remarked already in §2) the Lagrangian (2.2) is not of Liouville's form (21), so in polar coordinates—though they be the coordinates standard to the Keplerean literature—we cannot get to first base because we are not qualified even to play the game.

Looking finally to details associated with the adoption of confocal *parabolic* coordinates: we had

$$\begin{aligned} x &= \tfrac{1}{2}(\mu^2 - \nu^2) \\ y &= \mu\nu \end{aligned}$$

at (27), and working from (28) obtain

$$\begin{aligned} p_\mu &= m(\mu^2+\nu^2)\dot{\mu} \\ p_\nu &= m(\mu^2+\nu^2)\dot{\nu} \\ p_x &= \tfrac{1}{\mu^2+\nu^2}(\mu p_\mu - \nu p_\nu) \\ p_y &= \tfrac{1}{\mu^2+\nu^2}(\nu p_\mu + \mu p_\nu) \end{aligned}$$

while descriptions of $u_1(\mu)$, $u_2(\nu)$, $w_1(\mu)$ and $w_2(\nu)$ can be read off from (29). So we have[20]

$$G_{\text{parabolic}} = \tfrac{1}{2(\mu^2+\nu^2)}\Big\{(\mu^2 p_\nu^2 - \nu^2 p_\mu^2) - 2mk(\mu^2-\nu^2)\Big\} - \kappa \tag{50.1}$$

$$H = \tfrac{1}{\mu^2+\nu^2}\Big\{\tfrac{1}{2m}(p_\mu^2 + p_\nu^2) - 2k\Big\} \tag{50.2}$$

$$L_z = \tfrac{1}{2}(\mu p_\nu - \nu p_\nu) \tag{50.3}$$

$$K_x = \tfrac{1}{\mu^2+\nu^2}\Big\{\tfrac{1}{2m}(\mu^2 p_\nu^2 - \nu^2 p_\mu^2) - k(\mu^2-\nu^2)\Big\}$$

$$K_y = \tfrac{1}{\mu^2+\nu^2}\Big\{\tfrac{1}{2m}\big[\mu\nu(p_\mu^2 + p_\nu^2) - p_\mu p_\nu(\mu^2+\nu^2)\big] - 2k\mu\nu\Big\} \tag{50.4}$$

Evidently

$$G_{\text{parabolic}} = mK_x \tag{51}$$

where I have abandoned as an uninteresting triviality the additive constant which appears in (50.1).

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[19] See (54.1) in Reference 1.

[20] The last four of the following equations have been transcribed from (54.2) in the essay just cited.

The alternate polar and parabolic coordinate systems are in many respects complementary. The former led us to L_z, the latter to K_x. The former retains a floating constant but no directionality (isotropy emerged when the foci coalesced), the latter retains "floating directionality" but no adjustable constant (a has become infinite). At (51) the preferential reference to the x-component of $\boldsymbol{K}$ reflects our tacit decision at (27) to place the parabolic system in "standard position;" i.e., to remove the floating focus to the "negative end of the x-axis."

6. Identification of orbits with curves-of-constant-coordinate. I have several times remarked[21] that elliptical coordinate systems which share the property that they have one focus pinned at the force center give rise to populations of coordinate lines (curves on which a coordinate—be it ξ or η, s or θ, μ or ν—is constant) which invite interpretation as Keplerean orbits. I look now into some of the detailed ramifications of that elementary idea, and begin by assembling some familiar facts relating generally to the description of Keplerean orbits.

Let us agree to use the term "perihelion" when referring to

$$R \equiv \text{distance of closest approach to force center}$$

even though we do not imagine ourselves to be doing celestial mechanics, have not placed the literal "sun" at the force center, and anyway are discussing the *reduced* Kepler problem (imaginary "reduced mass" orbiting an imaginary force center). Speed v, distance r from the force center, and angular momentum ℓ stand in the especially simple relationship $\ell = mrv$ when the particle m crosses the principal axis (since it does so normally: $\boldsymbol{v} \perp \boldsymbol{r}$). From the energy relation

$$E = T + U = \tfrac{1}{2}\ell^2/mr^2 - k/r$$

we obtain

$$r^2 + \tfrac{k}{E}r - \ell^2/2mE = 0$$

This polynomial—first encountered in §1—has positive real roots (implying a bound orbit)

$$r = -\tfrac{k}{2E}\left[1 \pm \sqrt{1 + 2E\ell^2/mk^2}\right]$$

only if $-mk^2/2\ell^2 < E < 0$; in such (elliptic) cases we have

$$\text{perihelion } R = -\tfrac{k}{2E}\left[1 - \sqrt{1 + 2E\ell^2/mk^2}\right]$$
$$\text{aphelion} = -\tfrac{k}{2E}\left[1 + \sqrt{1 + 2E\ell^2/mk^2}\right]$$

giving

$$\text{semimajor axis} = \tfrac{1}{2}(\text{perihelion} + \text{aphelion}) = -\tfrac{k}{2E} \tag{52}$$

$$\text{eccentricity } e = \frac{\text{semimajor axis} - \text{perihelion}}{\text{semimajor axis}} = \sqrt{1 + 2E\ell^2/mk^2} \tag{53}$$

where the *ℓ-independence of the semimajor axis* merits special notice. In terms

[21] See again Figures 5 & 9 and associated text.

of the eccentricity one has

$$E = -\tfrac{mk^2}{2\ell^2}(1-e^2) \quad : \quad 0 \leqslant e < 1 \tag{54}$$

$$\begin{aligned} R &= -\tfrac{k}{2E}(1-e) \\ &= \tfrac{\ell^2}{mk}(1+e)^{-1} \end{aligned} \tag{55}$$

The Lentz vector $\boldsymbol{K}$, introduced at (38), has—since conserved—the same value at whatever orbital point it is evaluated, but is (by $\boldsymbol{v} \perp \boldsymbol{r}$) particularly easy to evaluate at perihelion; one has

$$\boldsymbol{K} = K\hat{\boldsymbol{R}} \quad \text{with} \quad \begin{aligned} K &= v\ell - k \\ &= \ell^2/mR - k \\ &= ke \end{aligned} \tag{56}$$

The Lenz vector $\boldsymbol{K}$ serves therefore to describe the *orientation and ellipticity* of the orbit. Specification of E and ℓ^2 are sufficient to determine the latter, but not the former. From $K = ke = k\sqrt{1 + 2E\ell^2/mk^2}$ we obtain

$$K^2 = k^2 + 2E\ell^2/m \tag{57}$$

so while one can specify $\hat{\boldsymbol{K}}$ arbitrarily the value of K is prefigured.

Occupying a special place within the population of elliptical orbits are the circular orbits; from the

$$\text{circularity condition: perihelion} = \text{aphelion}$$

we recover information

$$E = E_\odot = -mk^2/2\ell^2 \tag{58}$$

and

$$\begin{aligned} R = R_\odot &= -k/2E_\odot \\ &= \ell^2/mk \end{aligned} \tag{59}$$

reported already in §1. Circular orbits have eccentricity $e = 0$ and, since it is meaningless to speak of the "rotational orientation of a circle," we are not surprised to have $K = 0$.

In the limit $E \uparrow 0$ we obtain

$$\text{parihelion} \to \ell^2/2km \tag{60}$$

$$\begin{aligned} \text{aphelion} &\to \infty \ : \ \text{orbit becomes unbound (parabolic)} \\ \text{eccentricity} &\to 1 \\ K &\to k \end{aligned} \tag{61}$$

If $E > 0$ then

$$\begin{aligned} \text{perihelion } R = &\quad \tfrac{k}{2E}\Big[\sqrt{1+2E\ell^2/mk^2}-1\Big] \\ \text{“aphelion”} = &\underbrace{\tfrac{k}{2E}\Big[\sqrt{1+2E\ell^2/mk^2}+1\Big]}_{\text{refers to “remote branch” of hyperbola}} \end{aligned}$$

The “semimajor axis” retains its former meaning (distance between intercepts with principal axis) but acquires a modified description:

$$\text{semimajor axis} = \tfrac{1}{2}(\text{“aphelion”} - \text{periphelion}) = +\tfrac{k}{2E} \tag{62}$$

The definition of “eccentricity” is similarly adjusted:

$$\text{eccentricity } e = \frac{\text{semimajor axis + perihelion}}{\text{semimajor axis}} = \sqrt{1+2E\ell^2/mk^2}$$

One therefore has

$$E = +\tfrac{mk^2}{2\ell^2}(e^2-1) \quad : \quad e > 1 \tag{63}$$

$$\begin{aligned} R &= +\tfrac{k}{2E}(e-1) \\ &= \tfrac{\ell^2}{mk}(1+e)^{-1} \end{aligned} \tag{64}$$

which differ only in emphasis from their elliptic counterparts. And the equation $K = ke$ remains intact.

Thus reminded of the details by which physical/geometrical parameters enter into the design of Keplerean orbits, we are positioned to ask: Under what conditions can lines of coordinate constancy be associated with Keplerean orbits, and *vice versa*?

CIRCULAR ORBITS

Circular coordinate lines arise within the alternate polar system as lines of constant s:

$$s(\theta) = \text{constant}$$

Working from (25) we have

$$\begin{aligned} \frac{ds}{d\theta} &= \sqrt{\frac{E\cdot ma^2e^{2s} + kmae^s - \tfrac{1}{2}\ell^2}{\tfrac{1}{2}\ell^2}} \\ &= \sqrt{\frac{Er^2 + kr - \tfrac{1}{2m}\ell^2}{\tfrac{1}{2m}\ell^2}} \quad \text{with } \tfrac{1}{2}\ell^2 = \text{numerical value of } G_{\text{alternate polar}} \end{aligned}$$

so

$$\frac{ds}{d\theta} = 0 \quad \Rightarrow \quad r = \tfrac{k}{2E}\Big\{-1 \pm \sqrt{1+2E\ell^2/mk^2}\Big\}$$

We have recovered—now as an artifact of Liouville's method—precisely the equation upon which the review just ended was based. Circularity requires that the two roots be coincident, so we have

$$E_\odot = -mk^2/2\ell^2 < 0 \quad \text{and} \quad R_\odot = \ell^2/mk$$

—in precise agreement with (58) and (59). My intent here has been not so much (except as a check) to reproduce familiar results as to establish a pattern of argument.

RADIAL ORBITS

Free fall entails $\theta(s) = \text{constant}$ (i.e., $d\theta/ds = 0$). By adjustment of the details spelled out above it sets no condition on E, but requires $\ell = 0$.

ELLIPTIC ORBITS

Elliptic coordinate lines arise within $\{\xi, \eta\}$ systems as lines of constant ξ:

$$\xi(\eta) = \text{constant}; \quad \text{i.e.,} \quad \frac{d\xi}{d\eta} = 0$$

Arguing as before with the aid of (35) we have

$$\frac{d\xi}{d\eta} = \sqrt{\frac{E \cdot ma^2 \cosh^2\xi + kma\cosh\xi - \epsilon}{-E \cdot ma^2 \cos^2\eta + kma\cos\eta + \epsilon}}$$

$$\epsilon = \text{numerical value of } G_{\text{elliptic}}$$
$$= ma^2E + maK + \tfrac{1}{2}\ell^2$$

where I have allowed myself to drop the subscript from K_x. Evidently

$$a\cosh\xi = -\tfrac{k}{2E}\left\{1 \pm \sqrt{1 + 4E\epsilon/mk^2}\right\}$$

but this serves only to locate the greatest/least values of ξ encountered on an orbital tour; on a coordinate ellipse (ellipse of constant ξ) those are necessarily identical, so we have

$$E = -mk^2/4\epsilon < 0 \quad \text{and} \quad \underbrace{\langle a\cosh\xi\,\rangle}_{\text{denotes "constant value of..."}} = -k/2E = 2\epsilon/mk \tag{65}$$

Notice now that the ξ-ellipse—described parametrically by

$$x = a\cosh\xi\cos\eta - a$$
$$y = a\sinh\xi\sin\eta$$

—intercepts the x-axis at $\eta = 0$ (perihelion) and at $\eta = \pi$ (aphelion); the ξ-ellipse therefore has

$$\begin{aligned} \text{perihelion} &= a\cosh\xi - a \\ \text{aphelion} &= a\cosh\xi + a \\ \text{semimajor axis} &= a\cosh\xi \\ &= -\tfrac{k}{2E} \text{ by (65), consistently with (52)} \\ \text{ellipticity} &= 1/\cosh\xi \end{aligned}$$

Evidently

$$a = -\tfrac{k}{2E}e \tag{66}$$

which when introduced into $\epsilon = ma^2E + ma\cdot ke + \frac{1}{2}\ell^2$ gives

$$\epsilon = -\tfrac{mk^2}{4E}e^2 + \tfrac{1}{2}\ell^2 \tag{67}$$

On the other hand, (65) supplies

$$\epsilon = -\tfrac{mk^2}{4E}$$

so we recover (53): $e = \sqrt{1 + 2E\ell^2/mk^2}$. By slight rearrangement of the argument one could deduce that necessarily $K = ke$. The summary implication is that to describe a Keplerean ellipse as an ellipse of constant ξ (see again Figure 9) one has only to

- align the x-axis with the principal axis;
- set $\cosh\xi = 1/e$;
- set $a = -ke/2E$.

PARABOLIC ORBITS

All coordinate lines associated with the $\{\mu,\nu\}$ system are parabolic: ν-parameterized lines of constant μ open to the left, while μ-parameterized lines of constant ν open to the right, as illustrated in Figure 5. I elect to work with the former, writing

$$\mu(\nu) = \text{constant}$$

though the latter would serve as well. Arguing as before from (29) (in which I have without loss of generality set $\kappa = 0$) we have

$$\begin{aligned} \frac{d\mu}{d\nu} &= \sqrt{\frac{E\cdot m\mu^2 + mk - \epsilon}{E\cdot m\nu^2 + mk + \epsilon}} \\ \epsilon &= \text{numerical value of } G_{\text{parabolic}} \\ &= mK_x \quad \text{by (51)} \end{aligned}$$

Parabolic orbits arise if and only if $E = 0$, so from $d\mu/d\nu$=0 we obtain

$$K_x = k \quad \text{for parabolic orbits}$$

in precise agreement with (61). We obtain, however, no information about how the constant value of μ depends upon physical constants of the motion. But from $r = \frac{1}{2}(\mu^2 + \nu^2)$ we know that that $r = \frac{1}{2}\mu^2$ describes the μ-parabola's closest approach to the origin (see again Figure 5); we are let thus by (60) to the association

$$\mu\text{-parabola} \quad \longleftrightarrow \quad \text{Keplerean orbit with } E = 0,\ \ell^2 = km\mu^2,\ \boldsymbol{K} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$$

HYPERBOLIC ORBITS

Hyperbolic coordinate lines are presented by $\{\xi, \eta\}$ systems as lines of constant η:

$$\eta(\xi) = \text{constant}; \quad \text{i.e.,} \quad \frac{d\eta}{d\xi} = 0$$

We have physical interest only in the proximate branch ($\cos\eta > 0$: see again Figure 7) if the force is attractive ($k > 0$), which we will assume to be the case, and would have interest only in the remote branch in the contrary case. Adjustment of the argument employed in the elliptic case leads us to write

$$\frac{d\eta}{d\xi} = \sqrt{\frac{-E \cdot ma^2 \cos^2\eta \,+\, kma\cos\eta + \epsilon}{E \cdot ma^2 \cosh^2\xi + kma\cosh\xi - \epsilon}}$$

$$\begin{aligned} \epsilon &= \text{numerical value of } G_{\text{elliptic}} \\ &= ma^2E + maK + \tfrac{1}{2}\ell^2 \end{aligned}$$

Evidently

$$a\cos\eta = +\tfrac{k}{2E}\left\{1 \pm \sqrt{1 + 4E\epsilon/mk^2}\right\}$$

from which we obtain

$$E = -mk^2/4\epsilon > 0 \quad \text{and} \quad \langle a\cos\eta\,\rangle = +k/2E = -2\epsilon/mk \tag{68}$$

The proximate branch of the ξ-parameterized η-hyperbola intercepts the x-axis at $x = a\cos\eta - a < 0$, the remote branch at $x = -a\cos\eta - a \ll 0$, so we have

$$\begin{aligned} \text{perihelion} &= a - a\cos\eta \\ \text{“aphelion”} &= a + a\cos\eta \\ \text{semimajor axis} &= a\cos\eta \\ &= +\tfrac{k}{2E} \text{ by (68), consistently with (62)} \\ \text{ellipticity} &= 1/\cos\eta \end{aligned}$$

Evidently

$$a = +\tfrac{k}{2E}e \tag{69}$$

One final observation is required to bring the argument to completion: the vectors

$$\boldsymbol{a}_{\text{center}\,\rightarrow\,\text{focus}} \text{ and } \boldsymbol{K}_{\text{focus}\,\rightarrow\,\text{perihelion}} \text{ are } \begin{cases} \text{parallel in elliptic cases, but} \\ \text{antiparallel in hyperbolic cases} \end{cases}$$

It follows therefore from (44) that

$$\epsilon = \begin{cases} ma^2E + ma\cdot ke + \frac{1}{2}\ell^2 & \text{in elliptic cases, but} \\ \\ ma^2E - ma\cdot ke + \frac{1}{2}\ell^2 & \text{in hyperbolic cases} \end{cases}$$

Introducing (69) into the latter we again obtain $\epsilon = -\frac{mk^2}{4E}e^2 + \frac{1}{2}\ell^2$, and drawing upon (68) again recover (53): $e = \sqrt{1 + 2E\ell^2/mk^2}$. The summary implication is that to describe a Keplerean hyperbola as a hyperbola of constant $\cos\eta$ one has—in the attractive case—only to

- align the x-axis with the principal axis;
- set $\cos\eta = 1/e$;
- set $a = +ke/2E$.

By slight adjustment one could assign physicality alternatively to the remote branch, as required when the central force is repulsive ($k < 0$).

A claim make in text subsequent to Figure 9 is now substantiated: every Keplerean orbit can be displayed as a curve of the form

$$\text{elliptic coordinate} = \text{constant}$$

in some appropriately selected elliptic coordinate system (among which we have learned to include polar and parabolic systems as limiting cases). The argument has served to demonstrate the utility—at least in these simplest cases—of "Liouville's orbital equation" (first encountered at (25)), and has served also to deepen our understanding of "Liouville's observable" G.

Of course, Liouville's orbital equation serves in principle to describe *any* Keplerean orbit in *any* elliptic coordinate system, in which connection I borrow the following pretty argument from Goldstein: we have

$$\boldsymbol{K} \equiv \tfrac{1}{m}(\boldsymbol{p}\times\boldsymbol{L}) - \tfrac{k}{r}\boldsymbol{r}$$

giving

$$\begin{aligned} \boldsymbol{K}\cdot\boldsymbol{r} = Kr\cos\theta &= \tfrac{1}{m}\underbrace{\boldsymbol{r}\cdot(\boldsymbol{p}\times\boldsymbol{L})}_{=\,\boldsymbol{L}\cdot(\boldsymbol{r}\times\boldsymbol{p})\,=\,\ell^2} - kr \\ &\downarrow \\ \tfrac{1}{r} &= \tfrac{mk}{\ell^2}(1 + \tfrac{K}{k}\cos\theta) \\ &= \tfrac{mk}{\ell^2}(1 \pm e\cos\theta) \quad \text{with} \quad e = \sqrt{1 + \tfrac{2E\ell^2}{mk^2}} \end{aligned} \tag{70}$$

Familiarity with the Lenz vector $\boldsymbol{K}$ has led us here—swiftly and elegantly—to the "polar orbital equation" standard to the literature: (70) provides an explicit description of *all possible* Keplerean orbits.[22] By coordinate transformation

$$\{r, \theta\} \quad \longrightarrow \quad \{\text{elliptic coordinates}\}$$

it should, on this basis, be possible to describe *in all generality* the solutions of Liouville's equation... but I will not pursue this idea.

PART II: HAMILTON-JACOBI THEORY

7. Separation in spherical coordinates. Classical motion in any central force field is planar. Our work thus far contains no reference to the fact that the orbital plane lives in 3-space (or none beyond that implicit in the observation that Coulombic forces $F \sim r^{-2}$ become "geometrical" only in three dimensions). To look to the associated quantum theory—in effect, a theory of the "2-dimensional hydrogen atom"—is, however, to look to a "toy quantum theory," for the real hydrogen atom is a profoundly 3-dimensional object. Hamilton-Jacobi theory, though entirely classical (it contains no $\hbar$'s), is enough richer than the mechanics of Newton/Lagrange as to be susceptible to the force of a similar remark, and it is to be in better position to mark the respects in which 2-dimensional theory fails to reflect 3-dimensional reality that I step now into three dimensions, from which I will by "dimensional descent" soon parachute onto the orbital plane, taking careful note of the scenery lost from view in the process.

If spherical coordinates $\{r, \theta, \phi\}$ are introduced by this slight variant

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta\cos\phi \\ z &= r\sin\theta\sin\phi \end{aligned}\right\} \tag{71}$$

of the standard procedure[23] then

$$\begin{aligned} L &= \tfrac{1}{2}m[\,\dot{x}^2 + \dot{y}^2 + \dot{z}^2\,] + \tfrac{k}{r} \\ &= \tfrac{1}{2}m[\,\dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\phi}^2\,] + \tfrac{k}{r} \end{aligned}$$

gives

$$\begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \\ p_\phi &= m(r\sin\theta)^2\dot{\phi} \end{aligned}$$

[22] To ask on the basis of (70) for "orbits of constant r" is to recover (59).

[23] In (27) and (31) I honored convention by associating the line containing the foci with the x-axis. My 3-dimensional coordinate systems will be constructed by *revolution about the x-axis*; by this convention I become able to recover my 2-dimensional formulæ (i.e., to "turn off z) by the simple expedient of setting the revolution angle $\phi = 0$, but at this cost: the "polar axis"—standardly taken to be the z-axis—has become the x-axis.

whence[24]

$$H = \frac{1}{2m}\left[p_r^2 + \frac{1}{r^2}\,p_\theta^2 + \frac{1}{r^2\sin^2\theta}\,p_\phi^2\right] - \frac{k}{r} \tag{72}$$

The time-independent Hamilton-Jacobi equation reads

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{k}{r} = E \tag{73}$$

and on strength of the additivity assumption

$$S(r,\theta,\phi) = S_1(r) + S_2(\theta) + S_3(\phi) \tag{74}$$

separates; we obtain

$$\left.\begin{aligned} \frac{dS_3}{d\phi} &= \lambda_3 \\ \left(\frac{dS_2}{d\theta}\right)^2 + \frac{\lambda_3^2}{\sin^2\theta} &= \lambda_2^2 \\ \left(\frac{dS_1}{dr}\right)^2 + \frac{\lambda_2^2}{r^2} - 2m\left[\frac{k}{r} + E\right] &= 0 \end{aligned}\right\} \tag{75}$$

where λ_2 and λ_3 are separation constants (as also is E, which is an artifact of our having passed—by separation—from t-dependent Hamilton-Jacobi theory to its t-independent variant).

Had we, on the other hand, parachuted onto the orbital plane *prior* to invoking the apparatus of Hamilton-Jacobi theory—had we, in other words, elected to assign z the frozen value $z = 0$ ($\phi = 0$) and proceed from

$$H = \frac{1}{2m}\left[p_r^2 + \frac{1}{r^2}\,p_\theta^2\right] - \frac{k}{r} \tag{76}$$

—we would have obtained

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] - \frac{k}{r} = E$$

which by

$$S(r,\theta) = S_1(r) + S_2(\theta)$$

separates to become

$$\left.\begin{aligned} \left(\frac{dS_2}{d\theta}\right)^2 &= \lambda^2 \\ \left(\frac{dS_1}{dr}\right)^2 + \frac{\lambda^2}{r^2} - 2m\left[\frac{k}{r} + E\right] &= 0 \end{aligned}\right\} \tag{77}$$

which (in addition to E) contains only a single separation constant. What I meant by "scenery lost from view" becomes vividly evident when one compares (77) with (75).

[24] Compare M. Born, *The Mechanics of the Atom* (1924, English language edition of 1960), p. 132; G. Birtwhistle, *The Quantum Theory of the Atom* (1926), p. 187; Goldstein, p. 455.

For the moment I consider separation itself to be the point at issue, and am content therefore to postpone discussion of what the separated equations may be good for.[25]

8. Separation in alternate spherical coordinates. In place of (71) we write

$$\left.\begin{aligned} x &= ae^s\cos\theta \\ y &= ae^s\sin\theta\cos\phi \\ z &= ae^s\sin\theta\sin\phi \end{aligned}\right\} \tag{78}$$

Then

$$\begin{aligned} L &= \tfrac{1}{2}m[\,\dot{x}^2+\dot{y}^2+\dot{z}^2\,]+\tfrac{k}{r} \\ &= \tfrac{1}{2}ma^2e^{2s}[\,\dot{s}^2+\dot{\theta}^2+(\sin\theta)^2\dot{\phi}^2\,]+\tfrac{k}{a}e^{-s} \end{aligned} \tag{79}$$

gives

$$\begin{aligned} p_s &= ma^2e^{2s}\dot{s} \\ p_\theta &= ma^2e^{2s}\dot{\theta} \\ p_\phi &= ma^2e^{2s}\sin^2\theta\cdot\dot{\phi} \end{aligned}$$

whence

$$H=\frac{1}{2ma^2e^{2s}}\left[p_s^2+p_\theta^2+\frac{1}{\sin^2\theta}\,p_\phi^2\right]-\frac{k}{ae^s} \tag{80}$$

The associated Hamilton-Jacobi equation

$$\frac{1}{2ma^2e^{2s}}\left[\Big(\frac{\partial S}{\partial s}\Big)^2+\Big(\frac{\partial S}{\partial\theta}\Big)^2+\frac{1}{\sin^2\theta}\Big(\frac{\partial S_2}{\partial\phi}\Big)^2\right]-\frac{k}{ae^s}=E \tag{81}$$

separates by $S(s,\theta,\phi)=S_1(s)+S_2(\theta)+S_3(\phi)$ to give

$$\left.\begin{aligned} \frac{dS_3}{d\phi} &= \lambda_3 \\ \Big(\frac{dS_2}{d\theta}\Big)^2+\frac{\lambda_3^2}{\sin^2\theta} &= \lambda_2^2 \\ \Big(\frac{dS_1}{ds}\Big)^2+\lambda_2^2-2ma^2e^{2s}\Big[\frac{k}{ae^s}+E\Big] &= 0 \end{aligned}\right\} \tag{82}$$

which by $\frac{d}{ds}=r\frac{d}{dr}$ is readily seen to be equivalent to (75).

Had we elected to turn off z at the outset then—as was remarked already at (20)—the Lagrangian (79) would have assumed Liouville's design—a design *not* shared by (79). Separation proceeds in the standard way

[25] Born, Birtwhistle and Goldstein, in references just cited, explore that aspect of our subject in great detail.

$$\frac{1}{2ma^2e^{2s}}\left[\left(\frac{\partial S}{\partial s}\right)^2+\left(\frac{\partial S}{\partial \theta}\right)^2\right]-\frac{k}{ae^s}=E \tag{83}$$
$$\downarrow$$
$$\left.\begin{aligned}\left(\frac{dS_2}{d\theta}\right)^2&=\lambda^2\\ \left(\frac{dS_1}{ds}\right)^2+\lambda^2-2ma^2e^{2s}\Big[\frac{k}{ae^s}+E\Big]&=0\end{aligned}\right\} \tag{84}$$

which is gratifying, but the take-home lesson is this:

> Separability of the Hamilton-Jacobi equation does not presume Lagrangian separability in the sense of Liouville.

9. Separation in parabolic coordinates. The coordinate system now in question

$$\left.\begin{aligned}x&=\tfrac{1}{2}(\mu^2-\nu^2)\\ y&=\mu\nu\cos\phi\\ z&=\mu\nu\sin\phi\end{aligned}\right\} \tag{85}$$

gives back (27) at $\phi=0$, and should not be confused with the "paraboloidal coordinate system," which is something else.[26] From

$$\begin{aligned}L&=\tfrac{1}{2}m[\,\dot{x}^2+\dot{y}^2+\dot{z}^2\,]+\tfrac{k}{r}\\ &=\tfrac{1}{2}m\big[(\mu^2+\nu^2)(\dot{\mu}^2+\dot{\nu}^2)+\mu^2\nu^2\dot{\phi}^2\big]+\frac{2k}{\mu^2+\nu^2}\end{aligned}$$

—which, notably, does not display Liouville's design, but at $\phi=0$ gives back (28), which does—we are, following the pattern of previous argument, led to write

$$\begin{aligned}p_\mu&=m(\mu^2+\nu^2)\dot{\mu}\\ p_\nu&=m(\mu^2+\nu^2)\dot{\nu}\\ p_\phi&=m\mu^2\nu^2\dot{\phi}\end{aligned}$$

$$H=\frac{1}{2m}\left[\frac{1}{\mu^2+\nu^2}\{p_\mu^2+p_\nu^2\}+\frac{1}{\mu^2\nu^2}p_\phi^2\right]-\frac{2k}{\mu^2+\nu^2}$$

$$\frac{1}{2m}\left[\frac{1}{\mu^2+\nu^2}\left\{\left(\frac{\partial S}{\partial \mu}\right)^2+\left(\frac{\partial S}{\partial \nu}\right)^2\right\}+\frac{1}{\mu^2\nu^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right]-\frac{2k}{\mu^2+\nu^2}=E \tag{86}$$
$$\downarrow$$
$$\left.\begin{aligned}\left(\frac{dS_3}{d\phi}\right)&=\lambda_3\\ \left(\frac{dS_2}{d\nu}\right)^2+\frac{1}{\nu^2}\lambda_3^2-2m(k-E\nu^2)&=+\lambda_2\\ \left(\frac{dS_1}{d\mu}\right)^2+\frac{1}{\mu^2}\lambda_3^2-2m(k-E\mu^2)&=-\lambda_2\end{aligned}\right\} \tag{87}$$

[26] See Moon & Spencer, Reference 11, p. 44.

Had we, on the other hand, elected at the outset to work exclusively on the orbital plane $\phi = 0$ we would have been led to write

$$\frac{1}{2m}\left[\frac{1}{\mu^2+\nu^2}\left\{\left(\frac{\partial S}{\partial \mu}\right)^2+\left(\frac{\partial S}{\partial \nu}\right)^2\right\}\right]-\frac{2k}{\mu^2+\nu^2}=E \tag{88}$$

$$\downarrow$$

$$\left.\begin{aligned}\left(\frac{dS_2}{d\nu}\right)^2-2m(k+E\nu^2)&=+\lambda\\ \left(\frac{dS_1}{d\mu}\right)^2-2m(k+E\mu^2)&=-\lambda\end{aligned}\right\} \tag{89}$$

The separated equations obtained in this case are notable for their elegant symmetry.[27]

10. Separation in displaced prolate spheroidal coordinates. Prolate spheroidal coordinates

$$\begin{aligned}x&=a\cosh\xi\cos\eta\\ y&=a\sinh\xi\sin\eta\cos\phi\\ z&=a\sinh\xi\sin\eta\sin\phi\end{aligned}$$

arise when the confocal elliptic system (31) is revolved about the x-axis, and give back (31) at $\phi = 0$. The system is to be distinguished from the "ellipsoidal" and "conicoidal" systems, which are two quite different things,[28] and also from the "oblate spheroidal" system, which arises when (31) is revolved about the y-axis.[29] We have physical interest actually in this variant

$$\left.\begin{aligned}x&=a\cosh\xi\cos\eta-a\\ y&=a\sinh\xi\sin\eta\cos\phi\\ z&=a\sinh\xi\sin\eta\sin\phi\end{aligned}\right\} \tag{90}$$

where the displacement serves to place one focus at the Cartesian origin.

---

[27] Parabolic separation is discussed in §35 of Born, and in Birtwhistle's Chapter 9. Both authors cite P. Epstein and K. Schwartzschild, who in 1916 noticed independently that parabolic coordinates could be used (in the language provided by the "old quantum theory") to construct accounts of the Stark effect in hydrogen. Epstein and Schwartzschild took Jacobi's solution of Euler's "problem of two centers" as their point of departure, and by removing one force center to infinity (Jacobi's elliptic coordinates then become parabolic, as we have seen) synthesized the locally uniform electric field upon which the Stark effect depends. That parabolic coordinates figure also in Schrödinger's initial paper begins to seem not so amazing.

[28] See Moon & Spencer, Reference 11, pp. 37 & 40. Properties of prolate spheroidal coordinate systems are described on pp. 31–33.

[29] The foci are by this operation caused to trace a *circle* on the $\{x, z\}$-plane; oblate spheroidal coordinates might therefore prove useful in describing the motion of a particle in the presence of a *charged ring*.

Taking

$$\begin{aligned} L &= \tfrac{1}{2}m[\,\dot{x}^2+\dot{y}^2+\dot{z}^2\,]+\tfrac{k}{r} \\ &= \tfrac{1}{2}ma^2\big[(\cosh^2\xi-\cos^2\eta)(\dot{\xi}^2+\dot{\eta}^2)+2\sinh^2\xi\sin^2\eta\cdot\dot{\phi}^2\big]+\frac{k}{a(\cosh\xi-\cos\eta)} \end{aligned}$$

as our point of departure, we observe that we attain Liouville's design (34) only if ϕ is frozen ($\dot{\phi}=0$), and are led to write

$$\begin{aligned} p_\xi &= ma^2(\cosh^2\xi-\cos^2\eta)\,\dot{\xi} \\ p_\eta &= ma^2(\cosh^2\xi-\cos^2\eta)\,\dot{\eta} \\ p_\phi &= 2ma^2\sinh^2\xi\sin^2\eta\cdot\dot{\phi} \end{aligned}$$

The Hamilton, after a little manipulation, becomes

$$H=\frac{1}{2ma^2(\cosh^2\xi-\cos^2\eta)}\left\{\Big[p_\xi^2+p_\eta^2+2\Big(\frac{1}{\sinh^2\xi}+\frac{1}{\sin^2\eta}\Big)p_\phi^2\Big] + 2kma(\cosh\xi+\cos\eta)\right\}$$

giving

$$\frac{1}{2ma^2(\cosh^2\xi-\cos^2\eta)}\left\{\Big[\Big(\frac{\partial S}{\partial\xi}\Big)^2+\Big(\frac{\partial S}{\partial\eta}\Big)^2+2\Big(\frac{1}{\sinh^2\xi}+\frac{1}{\sin^2\eta}\Big)\Big(\frac{\partial S}{\partial\phi}\Big)^2\Big] + 2kma(\cosh\xi+\cos\eta)\right\}=E$$

whence

$$\left.\begin{aligned} \frac{dS_3}{d\phi} &= \lambda_3 \\ \Big(\frac{dS_2}{d\eta}\Big)^2+\frac{2\lambda_3^2}{\sin^2\eta}+2ma(k\cos\eta+Ea\cos^2\eta\,) &= +\lambda_2 \\ \Big(\frac{dS_1}{d\xi}\Big)^2+\frac{2\lambda_3^2}{\sinh^2\xi}+2ma(k\cosh\xi-Ea\cosh^2\xi) &= -\lambda_2 \end{aligned}\right\} \tag{91}$$

Prior abandonment of the third dimension would have resulted in the truncated system

$$\left.\begin{aligned} \Big(\frac{dS_2}{d\eta}\Big)^2+2ma(k\cos\eta+Ea\cos^2\eta\,) &= +\lambda_2 \\ \Big(\frac{dS_1}{d\xi}\Big)^2+2ma(k\cosh\xi-Ea\cosh^2\xi) &= -\lambda_2 \end{aligned}\right\} \tag{92}$$

All previously separated Hamilton-Jacobi systems can be recovered as limiting cases of (91) and (92).

Born, in his §39, provides a sketch of the Hamilton-Jacobi theory ("old quantum mechanics") of the "two-centers problem" and in that connection makes necessary use of what he calls "elliptic coordinates."[30] But neither Born nor Birtwhistle appears to have noticed (or at least to have attached any significance to the fact) that the Kepler problem—the "single center problem," posed as a problem in applied Hamilton-Jacobi theory—separates in (displaced) prolate spheroidal coordinates (irrespective of where the floating focus may have been placed). Goldstein (p. 457) remarks that "Separation of variables for the purely central force problem can also be performed in other coordinate systems, e.g., parabolic coordinates..." but gives no indication that he was, when he wrote, aware of the prolate spheroidal separability of the Kepler problem. Doubtless, the latter fact is (or once was) "well-known" to somebody, but I do not know who that somebody (Pauli?) may be or have been.[31]

Were a thorough account of the physical solution of the Kepler problem our objective, then (91) would represent not the end of the story, but only its beginning; ahead would lie discussion of

- the physical interpretations of the various separation constants;
- how one goes about recovering the theory of orbits;
- how one describes temporal progress along those orbits;
- implementation of Delaunay's theory of "action and angle variables"

—all good stuff, all susceptible to attack by methods Goldstein reviews in his Chapter 10, all of which I am content on this occasion to set aside.

11. Graphical display of some 3-dimensional coordinate systems. It is in an effort to make quite clear what we are talking about, and to lend visual variety to the presentation, that I insert here three figures representative of the coordinate systems which have recently generated so many equations. For the benefit of readers who may wish to engage in a bit of graphical experimentation (which I encourage—not least because the colored display is so pretty) I reproduce here the *Mathematica* code which produced Figure 10: after a precautionary `ClearAll` write

```
x[ξ_, η_, ϕ_] := Cosh[ξ]Cos[η] − 1
y[ξ_, η_, ϕ_] := Sinh[ξ]Sin[η]Cos[ϕ]
z[ξ_, η_, ϕ_] := Sinh[ξ]Sin[η]Sin[ϕ]
Ellipsoid=ParametricPlot3D[{x[1, η, ϕ], −y[1, η, ϕ], z[1, η, ϕ]},
                                                {η, 0, π}, {ϕ, π/2, 2π}]
Hyperboloid=ParametricPlot3D[{x[ξ, 3/4, ϕ], −y[ξ, 3/4, ϕ], z[ξ, 3/4, ϕ]},
                                                {ξ, 0, 3/2}, {ϕ, π/2, 2π}]
SpheroidalPlane=ParametricPlot3D[{x[ξ, η, π], −y[ξ, η, π], z[ξ, η, π]},
                                                {ξ, 0, 3/2}, {η, 0, π}]
```

[30] Born draws here upon work done by Pauli and—independently—by K. F. Niessen in 1922, and is content to be sketchy because that work produced numbers which we found to be in disagreement with H_2^+ data.

[31] Birtwhistle provides reference to Jacobi's *Vorlesungen über Dynamik*, p. 202, which might be the place to launch a serach of the literature.

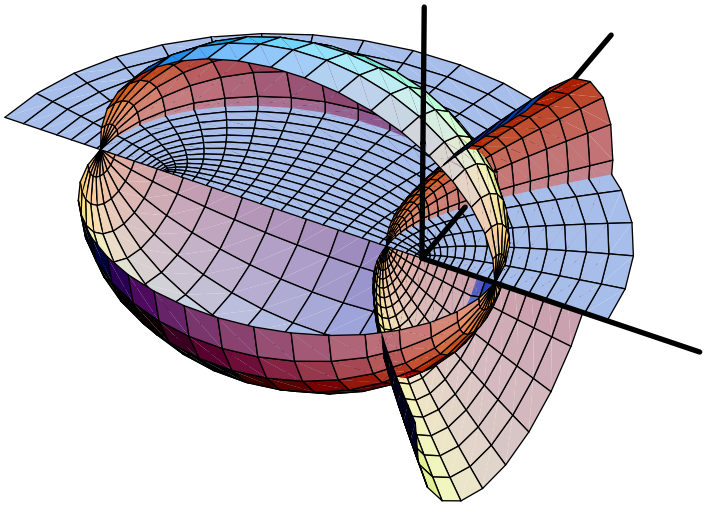

FIGURE 10: *Prolate spheroidal coordinate system (90), showing a*
- $\{\eta, \phi\}$*-coordinatized ellipsoid of constant* ξ*;*
- $\{\xi, \phi\}$*-coordinatized hyperbaloical sheet of constant* η*;*
- $\{\xi, \eta\}$*-coordinatized plane of constant* ϕ*.*

The right-handed Cartesian frame has been erected at what in the Keplerean application becomes the "physical focus," and the x*-axis has been identified with the focal axis.*

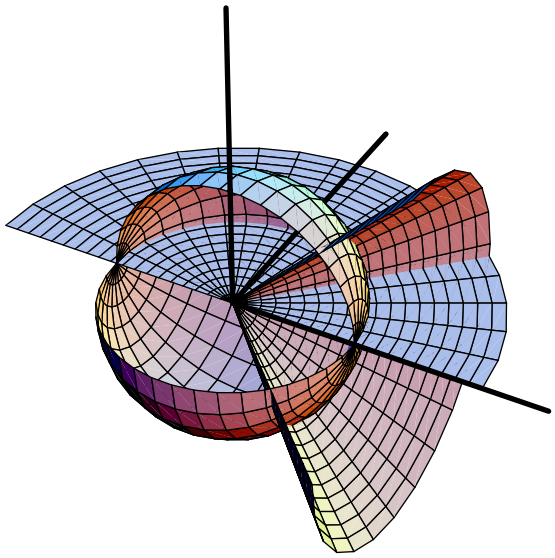

FIGURE 11: *(Alternate) spherical coordinate system (78), showing a*
- $\{\theta, \phi\}$*-coordinatized sphere of constant* $r \equiv ae^s$*;*
- $\{r, \phi\}$*-coordinatized cone of constant* θ*;*
- $\{r, \theta\}$*-coordinatized plane of constant* ϕ*.*

The Cartesian frame has been erected at the geometrical center, which in the Keplerean application becomes the "force center." The figure results from the preceding figure when the two foci coalesce, and illustrates why in the present context it is most natural to take the "polar axis" to be the x*-axis rather than (as is otherwise universal) the* z*-axis.*

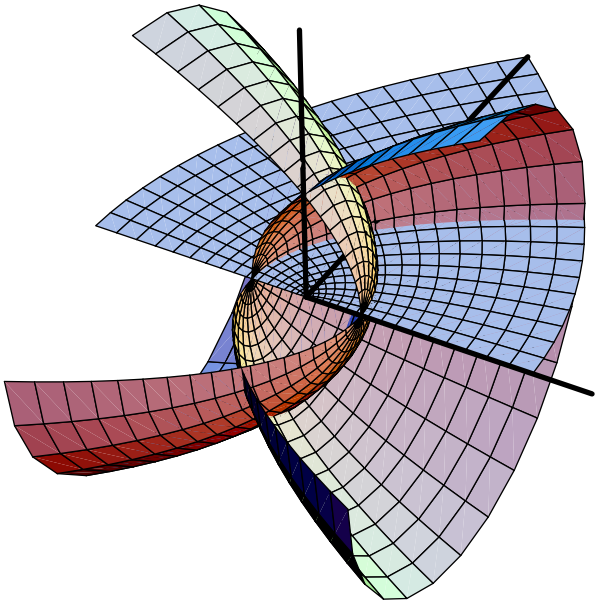

FIGURE 12: *Parabolic coordinate system (85), showing a*
- $\{\nu, \phi\}$*-coordinatized paraboloid of constant* μ *(opens left);*
- $\{\mu, \phi\}$*-coordinatized paraboloid of constant* ν *(opens right);*
- $\{\mu, \nu\}$*-coordinatized plane of constant* ϕ*.*

The Cartesian frame has been erected at the shared focus, which in the Keplerean application becomes the "force center." The figure results from Figure 10 when the "floating focus" is allowed to retreat to $x \to -\infty$*.*

```
X=Show[Graphics3D[{AbsoluteThickness[2],Line[{{0,0,0},{2,0,0}}]}]]
Y=Show[Graphics3D[{AbsoluteThickness[2],Line[{{0,0,0},{0,3,0}}]}]]
Z=Show[Graphics3D[{AbsoluteThickness[2],Line[{{0,0,0},{0,0,2}}]}]]
Show[{Ellipsoid,Hyperboloid,SpheroidalPlane,X,Y,Z},
                    PlotRange->All,Boxed->False,Axes->False]
```

Straightforward adjustment of that routine produced the other figures.

I have drawn attention in Figure 7 to the fact that in 2-dimensional confocal conic coordinate systems hyperbolas change their name $\eta \to 2\pi - \eta$ as they cross the line linking the foci, and to the fact that it is the value of $\cos\eta = \cos(2\pi - \eta)$ which serves more properly to distinguish one hyperbolic branch from another. It is notable, therefore, that in 3-dimensional prolate spheroidal systems each hyperboloidal sheet wears the same η identifier; the natural range of η is contracted from $[0, 2\pi]$ to $[0, \pi]$, and the former η-jump acquires the description $\phi \to \phi + \pi$.

Hyperbolic orbits refer, of course, to the physics of Keplerean scattering. If the central force is attractive ($k > 0$) then orbits can be inscribed on the η-sheet only if $0 \leqslant \eta < \frac{1}{2}\pi$ (similarly, $k < 0$ entails $\frac{1}{2}\pi < \eta \leqslant \pi$). Every given hyperbolic orbit can be inscribed on one of the hyperbolic sheets supplied by *some* suitably-selected prolate spheroidal system—the presumption being that the "floating focus" has been suitably placed. But with respect to any given prolate system most hyperbolic orbits will lie "off-sheet," and in the description of those one can expect η to make excursions outside the interval $[0, \frac{1}{2}\pi]$.

Somewhat relatedly: complexities of an entirely mathematical character—complexities having no counterpart in the physics of the matter—arise whenever an orbit passes through the point occupied by the "floating focus," but I have not the patience to spell out the details.

Finally I append a representation of the oblate spheroidal coordinate system, not because that system is useful in connection with the Kepler problem (it isn't, except in the spherical limit $a \downarrow 0$) but because it and its limiting form $a \uparrow \infty$ seem to me to be awaiting important assignments in other connections. It's an exceptionally pretty coordinate system, and deserves to be better known.

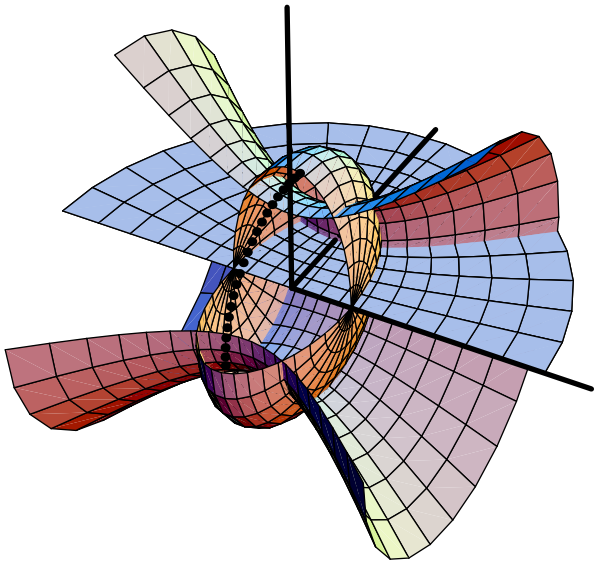

FIGURE 13: *Oblate spheroidal coordinate system*

$$x = a \sinh\xi \cos\eta$$
$$y = a \cosh\xi \sin\eta \cos\phi$$
$$z = a \cosh\xi \sin\eta \sin\phi$$

showing a

- $\{\eta, \phi\}$*-coordinatized oblate spheroid of constant* ξ*;*
- $\{\xi, \phi\}$*-coordinatized single-sheeted hyperboloid of constant* η*;*
- $\{\xi, \eta\}$*-coordinatized plane of constant* ϕ*.*

The Cartesian frame has been erected at the geometrical center. A circle of dots represents the "circle of foci." The coordinate system stands in no special relationship to the Kepler problem, though its plane sections (ϕ = *constant) are identical to those of the prolate spheroidal system; it has, I suspect, an important contribution to make to the analysis of some* other *problem.*

12. Universal Hamilton-Jacobi separability of Liouville's systems. We have seen by example that separability of the Hamilton-Jacobi equation does not presume separability in the sense of Liouville. I digress now to demonstrate that, on the other hand, systems of Liouville's design do universally give rise to separable Hamilton-Jacobi equations.

Lagrangians of the form

$$L = \tfrac{1}{2}\big[v_1(q_1)\dot{q}_1^2 + \cdots + v_n(q_n)\dot{q}_n^2\big] - \big[w_1(q_1) + \cdots + w_n(q_n)\big]$$

lead trivially to uncoupled equations of motion. Liouville (1849) explored implications of the less restrictive assumption

$$\begin{aligned} L &= \tfrac{1}{2}u\big[v_1(q_1)\dot{q}_1^2 + \cdots + v_n(q_n)\dot{q}_n^2\big] - \frac{\big[w_1(q_1) + \cdots + w_n(q_n)\big]}{u} \\ u &\equiv u_1(q_1) + \cdots + u_n(q_n) \end{aligned}$$

which by a change of variable[32] is tantamount to an assumption that the Lagrangian possesses what I have called "Liouville's design"[33]

$$\left.\begin{aligned} L &= \tfrac{1}{2}u\big[\dot{q}_1^2 + \cdots + \dot{q}_n^2\big] - \frac{\big[w_1(q_1) + \cdots + w_n(q_n)\big]}{u} \\ u &\equiv u_1(q_1) + \cdots + u_n(q_n) \end{aligned}\right\} \tag{93}$$

From $p_1 = u\dot{q}_1, \ldots, p_n = u\dot{q}_n$ we are led to a Hamiltonian of the correspondingly distinctive but even simpler design

$$H = \tfrac{1}{u}\Big\{\tfrac{1}{2}\big[p_1^2 + \cdots + p_n^2\big] + \big[w_1(q_1) + \cdots + w_n(q_n)\big]\Big\} \tag{94}$$

The associated time-independent Hamilton-Jacobi equation

$$\tfrac{1}{2}\big[\big(\tfrac{\partial S}{\partial q_1}\big)^2 + \cdots + \big(\tfrac{\partial S}{\partial q_n}\big)^2\big] + \big[w_1(q_1) + \cdots + w_n(q_n)\big] = E\,u \tag{95}$$

immediately separates on strength of the assumption

$$S(q_1, \ldots, q_n) = S_1(q_1) + \cdots + S_n(q_n)$$

to yield the following uncoupled system of ordinary differential equations

$$\left.\begin{aligned} \tfrac{1}{2}\big(\tfrac{dS_1}{dq_1}\big)^2 + \big[w_1(q_1) - E\,u_1(q_1)\big] &= \epsilon_1 \\ \tfrac{1}{2}\big(\tfrac{dS_2}{dq_2}\big)^2 + \big[w_2(q_2) - E\,u_2(q_2)\big] &= \epsilon_2 \\ &\vdots \\ \tfrac{1}{2}\big(\tfrac{dS_n}{dq_n}\big)^2 + \big[w_n(q_n) - E\,u_n(q_n)\big] &= \epsilon_n \end{aligned}\right\} \tag{96}$$

[32] See §43 in E. T. Whittaker's *Analytical Dynamics* (1937) for all missing details.

[33] Notice that Liouville's design requires the variables q to be co-dimensional.

where the (necessarily co-dimensional) separation constants $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ are subject necessarily to the following constraint:

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0 \tag{97}$$

The "universal separability" to which my section title refers is thus established; what follows is by nature commentary.

Liouville himself, by Whittaker's account, was interested not in separation of the Hamilton-Jacobi equation but in decoupling the equations of motion—which he did not quite accomplish (except in a familiar special case). By clever analysis[34] he proceeded from the Lagrange equations implicit in (93) to a system of first-order differential equations

$$\left.\begin{array}{l} \frac{1}{2}u^2\cdot\dot{q}_1^2 + \big[w_1(q_1) - E\,u_1(q_1)\big] = \epsilon_1 \\ \frac{1}{2}u^2\cdot\dot{q}_2^2 + \big[w_2(q_2) - E\,u_2(q_2)\big] = \epsilon_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \frac{1}{2}u^2\cdot\dot{q}_n^2 + \big[w_n(q_n) - E\,u_n(q_n)\big] = \epsilon_n \end{array}\right\} \tag{98}$$

which by $u\cdot\dot{q}_k = p_k = \frac{\partial S}{\partial q_k} = \frac{\partial S_k}{\partial q_k}$ are in fact precisely equivalent to (96). But while the system (96) is assuredly/universally separated, the system (98) becomes uncoupled only in the case $u =$ constant, and in that case the Lagrange equations latent in (93) are *automatically* uncoupled: Liouville can in that case claim simply to have *reduced the order* of the system.

In the case $n = 2$ Liouville eliminates the offending u^2 term by dividing one equation—actually its square root—into the other; that procedure serves to eliminate not only the offending u-factor but also all reference to motion! One obtains an equation of the form

$$\frac{dq_1}{dq_2} = \sqrt{\frac{E\cdot u_1(q_1) - w_1(q_1) - \epsilon}{E\cdot u_2(q_2) - w_2(q_2) + \epsilon}}$$

which we first encountered at (25), an equation which refers through $q_1(q_2)$ to the *design of the orbit*; information relating to *rate of progress along that orbit* has then to be extracted from energy conservation by a separate argument. In higher-dimensional cases one might write

$$q_i = q_i(q_n) \quad : \quad i = 1, 2, \ldots, n-1$$

and proceed similarly... into the higher reaches of analytical intractability.

A paraphrase of Liouville's analysis could be erected upon the base provided by the canonical equations latent in (94)

$$\begin{array}{ll} \dot{q}_1 = \frac{1}{u}p_1 & \dot{p}_1 = \frac{1}{u}\big[H\,u_1' - w_1'\big]\dot{u}_1 \\ \dot{q}_2 = \frac{1}{u}p_2 & \dot{p}_2 = \frac{1}{u}\big[H\,u_2' - w_2'\big]\dot{u}_2 \\ \quad\vdots & \quad\vdots \\ \dot{q}_n = \frac{1}{u}p_n & \dot{p}_n = \frac{1}{u}\big[H\,u_n' - w_n'\big]\dot{u}_n \end{array}$$

[34] See Whittaker, or §3 in Reference 1.

but I will not belabor the details.

Suppose $\{q_1, q_2, q_3\}$ refer—relative to an inertial Cartesian frame—to a locally orthogonal coordinatization of physical 3-space:

$$(d\sigma)^2 \equiv (dx)^2 + (dy)^2 + (dz)^2 = g_1(dq_1)^2 + g_2(dq_2)^2 + g_3(dq_3)^2$$

Then

$$\begin{aligned} L &= \tfrac{1}{2}m\big[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big] - V(x,y,z) \\ &= \tfrac{1}{2}m\big[g_1\dot{q}_1^2 + g_2\dot{q}_2^2 + g_3\dot{q}_3^2\big] - U(q_1,q_2,q_3) \end{aligned}$$

will possess Liouville's design if an only if

$$g_1 = g_2 = g_3 = u(q_1,q_2,q_3) \equiv u_1(q_1) + u_2(q_2) + u_3(q_3) \tag{99.1}$$

and (simultaneously) the potential energy function has acquired the structure

$$U(q_1,q_2,q_3) = \frac{w_1(q_1) + w_2(q_2) + w_3(q_3)}{u(q_1,q_2,q_3)} \tag{99.2}$$

In the 3-dimensional coordinate systems in which we have acquired interest one has[35]

$$\begin{aligned} \text{SPHERICAL}: \quad & (d\sigma)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2 \\ \text{ALTERNATE SPHERICAL}: \quad & (d\sigma)^2 = a^2e^{2s}\big[(ds)^2 + (d\theta)^2\big] + a^2e^{2s}\sin^2\theta(d\phi)^2 \\ \text{PARABOLIC}: \quad & (d\sigma)^2 = (\mu^2+\nu^2)\big[(d\mu)^2 + (d\nu)^2\big] + \mu^2\nu^2(d\phi)^2 \\ \text{PROLATE SPHEROIDAL}: \quad & (d\sigma)^2 = a^2(\cosh^2\xi + \cos^2\eta)\big[(d\xi)^2 + (d\eta)^2\big] \\ & \qquad + a^2\sinh^2\xi\sin^2\eta(d\phi)^2 \end{aligned}$$

—*none* of which satisfy condition (99.1); Hamilton-Jacobi separability in those cases is (as previously noted) "non-Liouvillian." But if one extinguishes the third dimension ($d\phi = 0$) then *all but the spherical (polar) system* conforms to (99.1), and (as we have seen) the Keplerean potential conforms also to (99.2); in those cases separability "on the orbital plane" can be said to be "Liouvillian." Some the preceding details will acquire new interest when, in connection with separation of the Keplerean Schrödinger equation, we undertake to construct curvilinear generalizations of ∇^2.

On several occasions subsequent to (37) we were able to obtain valuable information by the simple expedient of "promoting Liouville's separation constant to the status of an observable." I explore now the higher-dimensional ramifications of that idea, which—working from (96)—becomes

$$\left.\begin{aligned} \epsilon_1 \to G_1 &\equiv \tfrac{1}{2}p_1^2 + \big[w_1(q_1) - H\,u_1(q_1)\big] \\ \epsilon_2 \to G_2 &\equiv \tfrac{1}{2}p_2^2 + \big[w_2(q_2) - H\,u_2(q_2)\big] \\ &\vdots \\ \epsilon_n \to G_n &\equiv \tfrac{1}{2}p_n^2 + \big[w_n(q_n) - H\,u_n(q_n)\big] \end{aligned}\right\} \tag{100}$$

[35] I take my information from Moon & Spencer.

where H was described at (94). I will, as a matter of expository convenience, henceforth set $n = 3$. The point transformation

$$\{x, y, z\} \longrightarrow \{q_1, q_2, q_3\}$$

sends $T = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2] \longrightarrow u[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2]$, and in doing so induces the "extended point transformation"

$$\{x, y, z; p_x, p_y, p_z\} \longrightarrow \{q_1, q_2, q_3; p_1, p_2, p_3\}$$

Since such transformations are necessarily canonical, it becomes a matter of indifference whether we use Cartesian or curvilinear variables when computing Poisson brackets, and—since it was in terms of the latter that the observables G presented themselves at (100)—it proves more natural to use curvilinear coordinates. With the (heroic!) assistance of *Mathematica* we quickly compute

$$[H, G_1] = [H, G_2] = [H, G_3] = 0 \tag{101}$$

and verify that

$$G_1 + G_2 + G_3 = 0 \tag{102}$$

Moreover

$$[G_1, G_2] = [G_1, G_3] = [G_2, G_3] = 0 \tag{103}$$

We expect these results to sustain $3 \to n$ generalization; I have not attempted to construct a formal proof, but will proceed as though such a proof were in hand.

I know of no "natural" way to select or assemble $n-1$ objects from n, so do the obvious simple (but symmetry-breaking) thing: I select $\{G_1, G_2, \ldots, G_{n-1}\}$ and understand G_n to be recoverable from $G_n = -(G_1 + G_2 + \cdots + G_{n-1})$. The set $\{H, G_1, G_2, \ldots, G_{n-1}\}$ is a set of n observables, each of which (in the Poisson bracket sense) commutes with every other; the set is, in other words, *involutory* (its elements are "in involution"), with powerful consequences (some due to Liouville himself) which are detailed in Whittaker's §§147 & 148.[36]

It is a notable fact that the conserved observables G_i supplied by (100) *are of non-Noetherean design*. The substance of this remark has been detailed elsewhere,[37] but its deeper meaning and ultimate consequences I do not yet understand.

In the case $n = 2$ we are at (100) supplied with only a single observable, namely

$$G_1 \equiv \tfrac{1}{2}p_1^2 + \big[w_1(q_1) - H\, u_1(q_1)\big] \tag{104}$$

[36] In quantum mechanics such maximal involutory sets become "complete sets of simultaneous (or commuting) observables."

[37] See §4 in Reference 1. In essence, the observables J supplied by Noether's theorem acquire their quadratic p-dependence entirely from the Hamiltonian, which enters as a factor, but the observables G generally exhibit also some "dangling p-dependence."

which on its face does not much resemble either $-G_2$ or the G which at (37) was obtained by an "E-elimination procedure" and gave rise ultimately to (42). But when written out in explicit detail, (104) reads

$$G_1 = \frac{\big[u_2(q_2)p_1^2 - u_1(q_1)p_2^2\big] + 2\big[u_2(q_2)w_1(q_1) - u_1(q_1)w_2(q_2)\big]}{2\big[u_1(q_1) + u_2(q_2)\big]}$$

which clearly does conform to (102)

$$G_1 + G_2 = 0$$

and clearly does mimic the design of (37).

I bring this discussion to an arbitrary end with some brief historical and bibliographic remarks. Whittaker concludes his §43 with reference to the "further investigations on this subject" by P. Stäckel (1893), J. Hadamard (1911), P. Burgatti (1911) and V. S. Vrkljan (1930). From the more detailed references to the work of Stäckel[38] supplied by Moon & Spencer it becomes apparent that Stäckel drew his motivation directly from Hamilton-Jacobi theory (not from the theory of partial differential equations in some more generalized sense), while the references to Stäckel which Goldstein introduced into his second edition (see especially p. 453 and Appendix D) make it abundantly clear that Stäckel took Liouville's work as his specific point of departure.[39]

PART III: SCHRÖDINGER THEORY

13. Escape from Cartesian tyranny: curvilinear quantization. Though Newton's creative fire took its energy from sources which will always remain somewhat wrapped in mystery, it is clear that he worked always in reaction to the metaphysics of René Descartes, and it becomes therefore one of the ironies of history that his first act in the *Principia* (1687) was to attach a Cartesian frame to the fixed stars, and thus to place mechanics under the thumb of a kind of "Cartesian tyranny." Autocratic regimes may simplify life, but provide limited opportunity, and have ultimately to be cast off. Which may take awhile. In mechanics the escape from Cartesian tyranny—of which

$$\boldsymbol{F} = m\ddot{\boldsymbol{x}} \quad \longrightarrow \quad \left\{\tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}} - \tfrac{\partial}{\partial q}\right\}L = 0$$

[38] "Über die Integration der Hamilton-Jacobischen Differentialgleichung mittels Separation der Variabelen" (1891); "Sur une classe de problèmes de dynamique" (1893); "Über die Integration der Hamiltonschen Differentialgleichung mittels Separation der Variabelen" (1897).

[39] The *Dictionary of Scientific Biography* accords Paul Stäckel (1862–1919) only brief notice: we are informed that he was a disciple of Weierstrass, taught at Heidelberg, specialized in analytical mechanics and related geometry, the theory of analytic functions and (in later years) set theory and the theory of prime numbers. He was also active as an historian of 18$^{\text{th}}$ and early 19$^{\text{th}}$ Century science; he played a major role in arranging for publication of Euler's *Opera omnia* and also edited works of (among others) Gauss and Jacobi.

provides a cartoon—took the better part of a century, and became complete only with the publication (1788) of Lagrange's *Mechanique analytique*.

Quantum mechanics admits of famously many distinct but equivalent formulations, of which some are better known than others, but all (or almost all—certainly all of the *standard* formulations) were created by, and are today cultivated by, physicists seemingly content to go about their practical work subject to the strictures of a benign Cartesian tyranny. Of present interest to me is the formalism formerly known as "wave mechanics," but which I now find it convenient to call "Schrödinger theory." Within that formalism one proceeds

$$\begin{array}{ccc} \text{Hamiltonian}\big|_{\text{Cartesian}} & \longrightarrow & \text{Hamiltonian}\big|_{\text{curvilinear}} \\ \Big\downarrow\ p \to \frac{\hbar}{i}\frac{\partial}{\partial x} & & \\ \text{Schrödinger}\big|_{\text{Cartesian}} & \longrightarrow & \text{Schrödinger}\big|_{\text{curvilinear}} \end{array}$$

and the subtle presence of Cartesian tyranny (or is it merely chauvinism?) is revealed the absence of a descending arrow on the right. Such an arrow *can* be installed, but it takes some doing.[40] But since it leads to a result which (at least in flat space) is not in dispute, I will—with explicit acknowledgement that I am doing so—be content on this occasion to join the masses in acquiescing to the now-traditional tyranny.

So we take

$$\left\{-\tfrac{\hbar^2}{2m}\nabla^2 - k\frac{1}{\sqrt{x^2+y^2+z^2}}\right\}\psi = E\psi \tag{105}$$

as our point of departure, and recall that if curvilinear coordinates $\{q^1, q^2, q^3\}$ are introduced by equations of the form

$$\begin{aligned} x &= x(q^1, q^2, q^3) \\ y &= y(q^1, q^2, q^3) \\ z &= z(q^1, q^2, q^3) \end{aligned}$$

then

$$(ds)^2 \equiv (dx)^2 + (dy)^2 + (dz)^2 = g_{ij}(q)dq^i dq^j$$

[40] One must recognize first of all that the wave function $\psi(\boldsymbol{x})$ transforms not as a scalar field, but as a *scalar density of weight* $W = \frac{1}{2}$. One must recognize also that (and how) Laplace-Beltrami operators $\nabla^2(W)$ appropriate to scalar densities are to be distinguished from the (more familiar) operators $\nabla^2 \equiv \nabla^2(0)$ appropriate to simple scalars. One then finds that the quantization procedure $\boldsymbol{p} \to \frac{\hbar}{i}\boldsymbol{\nabla}$ has in general to be described

$$\boldsymbol{p} \longrightarrow \text{covariant derivative operator of weight } W = \tfrac{1}{2}$$

For an elaborate account of the details (the overthrow of tyranny sometimes calls for terrorism) see QUANTUM MECHANICS (1967), Chapter 2, pp. 153–183.

where the metric matrix $\|g_{ij}\|$ is everywhere diagonal if the q-system is (everywhere locally) orthogonal. In all events one has

$$\nabla^2 = \tfrac{1}{\sqrt{g}}\tfrac{\partial}{\partial q^i}\sqrt{g}g^{ij}\tfrac{\partial}{\partial q^j}$$

(here $\|g^{ij}\| \equiv \|g_{ij}\|^{-1}$ and $g \equiv \det\|g_{ij}\|$) which in orthogonal cases simplifies:

$$\nabla^2 = \tfrac{1}{\sqrt{g}}\tfrac{\partial}{\partial q^1}\sqrt{g}g^{11}\tfrac{\partial}{\partial q^1} + \tfrac{1}{\sqrt{g}}\tfrac{\partial}{\partial q^2}\sqrt{g}g^{22}\tfrac{\partial}{\partial q^2} + \tfrac{1}{\sqrt{g}}\tfrac{\partial}{\partial q^3}\sqrt{g}g^{33}\tfrac{\partial}{\partial q^3}$$

And since

$$\|g_{ij}\| = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} \text{ entails } g = g_1g_2g_3 \text{ and } \|g^{ij}\| = \begin{pmatrix} g_1^{-1} & 0 & 0 \\ 0 & g_2^{-1} & 0 \\ 0 & 0 & g_3^{-1} \end{pmatrix}$$

we have

$$\nabla^2 = \tfrac{1}{\sqrt{g_1g_2g_3}}\left\{\tfrac{\partial}{\partial q^1}\sqrt{\tfrac{g_2g_3}{g_1}}\tfrac{\partial}{\partial q^1} + \tfrac{\partial}{\partial q^2}\sqrt{\tfrac{g_1g_3}{g_2}}\tfrac{\partial}{\partial q^2} + \tfrac{\partial}{\partial q^3}\sqrt{\tfrac{g_1g_2}{g_3}}\tfrac{\partial}{\partial q^3}\right\}$$

so, taking our metric data from equations displayed just prior to (100), we obtain

$$\nabla^2_{\text{Cartesian}} = \left(\tfrac{\partial}{\partial x}\right)^2 + \left(\tfrac{\partial}{\partial y}\right)^2 + \left(\tfrac{\partial}{\partial z}\right)^2 \tag{106.1}$$

$$\nabla^2_{\text{spherical}} = \tfrac{1}{r^2\sin\theta}\left\{\tfrac{\partial}{\partial r}r^2\sin\theta\tfrac{\partial}{\partial r} + \tfrac{\partial}{\partial\theta}\sin\theta\tfrac{\partial}{\partial\theta} + \tfrac{\partial}{\partial\phi}\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\phi}\right\} \tag{106.2}$$

$$\nabla^2_{\text{alternate}} = \tfrac{1}{a^3e^{3s}\sin\theta}\left\{\tfrac{\partial}{\partial s}ae^s\sin\theta\tfrac{\partial}{\partial s} + \tfrac{\partial}{\partial\theta}ae^s\sin\theta\tfrac{\partial}{\partial\theta} + \tfrac{\partial}{\partial\phi}\tfrac{ae^s}{\sin\theta}\tfrac{\partial}{\partial\phi}\right\} \tag{106.3}$$

$$\nabla^2_{\text{parabolic}} = \tfrac{1}{\mu\nu(\mu^2+\nu^2)}\left\{\tfrac{\partial}{\partial\mu}\mu\nu\tfrac{\partial}{\partial\mu} + \tfrac{\partial}{\partial\nu}\mu\nu\tfrac{\partial}{\partial\nu} + \tfrac{\partial}{\partial\phi}\tfrac{\mu^2+\nu^2}{\mu\nu}\tfrac{\partial}{\partial\phi}\right\} \tag{106.4}$$

$$\nabla^2_{\text{displaced spheroidal}} = \tfrac{1}{a^3\sinh\xi\sin\eta(\sinh^2\xi+\sin^2\eta)}\left\{\tfrac{\partial}{\partial\xi}a\sinh\xi\sin\eta\tfrac{\partial}{\partial\xi} + \tfrac{\partial}{\partial\eta}a\sinh\xi\sin\eta\tfrac{\partial}{\partial\eta} + \tfrac{\partial}{\partial\phi}\tfrac{\sinh^2\xi+\sin^2\eta}{\sinh\xi\sin\eta}\tfrac{\partial}{\partial\phi}\right\} \tag{106.5}$$

The Kepler problem requires us to recall also these implications

$$\begin{aligned}\sqrt{x^2+y^2+z^2} &= r \\ &= ae^s \\ &= \tfrac{1}{2}(\mu^2+\nu^2) \\ &= a(\cosh\xi - \cos\eta)\end{aligned}$$

of the coordinate definitions (71),(78), (85) and (90), and in connection with (106.5) it proves useful to notice that $\sinh^2\xi + \sin^2\eta = \cosh^2\xi - \cos^2\eta$.

Pull-back to the orbital plane—identified with the $\{x, y\}$-plane—leaves the classical physics (though not the classical Hamilton-Jacobi physics) of central

force problems unaffected, but is profoundly unphysical in quantum mechanics, where it becomes a merely formal exercise. But informative, for that very reason. In preparation for such exercise we use

$$\nabla^2 = \tfrac{1}{\sqrt{g_1 g_2}}\left\{ \tfrac{\partial}{\partial q^1}\sqrt{\tfrac{g_2}{g_1}}\tfrac{\partial}{\partial q^1} + \tfrac{\partial}{\partial q^2}\sqrt{\tfrac{g_1}{g_2}}\tfrac{\partial}{\partial q^2}\right\} \tag{107}$$

to obtain

$$\nabla^2_{\text{Cartesian}} = \left(\tfrac{\partial}{\partial x}\right)^2 + \left(\tfrac{\partial}{\partial y}\right)^2 \tag{108.1}$$

$$\nabla^2_{\text{polar}} = \tfrac{1}{r}\left\{ \tfrac{\partial}{\partial r} r \tfrac{\partial}{\partial r} + \tfrac{\partial}{\partial \theta}\tfrac{1}{r}\tfrac{\partial}{\partial \theta}\right\} \tag{108.2}$$

$$\nabla^2_{\text{alternate}} = \tfrac{1}{a^2 e^{2s}}\left\{ \left(\tfrac{\partial}{\partial s}\right)^2 + \left(\tfrac{\partial}{\partial \theta}\right)^2\right\} \tag{108.3}$$

$$\nabla^2_{\text{parabolic}} = \tfrac{1}{\mu^2+\nu^2}\left\{ \left(\tfrac{\partial}{\partial \mu}\right)^2 + \left(\tfrac{\partial}{\partial \nu}\right)^2\right\} \tag{108.4}$$

$$\nabla^2_{\text{displaced conic}} = \tfrac{1}{a^2(\sinh^2\xi + \sin^2\eta)}\left\{ \left(\tfrac{\partial}{\partial \xi}\right)^2 + \left(\tfrac{\partial}{\partial \eta}\right)^2\right\} \tag{108.5}$$

Note how non-trivial is the procedure by which (107) is extracted from its 3-dimensional counterpart; i.e., by which (106) → (108). And how especially simple (except in the polar case) are the latter equations. That simplicity is a direct reflection of the circumstance that the metric is in those cases "conformally Euclidean"

$$\|g_{ij}\| = u \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and endowed therefore with the specialized structure assumed by Liouville.

Chapter 21 of Abramowitz & Stegun's *Handbook of Mathematical Functions* provides a valuable account of "Spheroidal Wave Functions," but proceeds from definitions of the associated prolate/oblate coordinate systems which depart from my own conventions. In essence, where I (at the beginning of §9) wrote

$$\begin{aligned} x &= a\cosh\xi\cos\eta \\ y &= a\sinh\xi\sin\eta\cos\phi \\ z &= a\sinh\xi\sin\eta\sin\phi \end{aligned}$$

they would write (but without the hats)

$$\begin{aligned} x &= a\hat{\xi}\,\hat{\eta} \\ y &= a\sqrt{(\hat{\xi}^2-1)(1-\hat{\eta}^2)}\cos\phi \\ z &= a\sqrt{(\hat{\xi}^2-1)(1-\hat{\eta}^2)}\sin\phi \end{aligned}$$

and in place of our "displaced" system (90) they would write

$$\left.\begin{aligned} x &= a\hat{\xi}\,\hat{\eta} - a \\ y &= a\sqrt{(\hat{\xi}^2-1)(1-\hat{\eta}^2)}\cos\phi \\ z &= a\sqrt{(\hat{\xi}^2-1)(1-\hat{\eta}^2)}\sin\phi \end{aligned}\right\} \tag{109}$$

One in either case (with or without displacement) computes

$$(ds)^2 = a^2\,\tfrac{\hat\xi^2-\hat\eta^2}{\hat\xi^2-1}(d\xi)^2 + a^2\,\tfrac{\hat\xi^2-\hat\eta^2}{1\ -\hat\eta^2}(d\eta)^2 + a^2(\hat\xi^2-1)(1-\hat\eta^2)(d\phi)^2$$

giving

$$\nabla^2 = \tfrac{1}{a^3(\hat\xi^2-\hat\eta^2)}\Big\{\tfrac{\partial}{\partial\hat\xi}\,a(\hat\xi^2-1)\,\tfrac{\partial}{\partial\hat\xi} + \tfrac{\partial}{\partial\hat\eta}\,a(1-\hat\eta^2)\,\tfrac{\partial}{\partial\hat\eta} + \tfrac{\partial}{\partial\phi}\,a\,\tfrac{\hat\xi^2-\hat\eta^2}{(\hat\xi^2-1)(1-\hat\eta^2)}\,\tfrac{\partial}{\partial\phi}\Big\} \tag{110}$$

"Displacement" does, however, serve to simplify the description of

$$r = \sqrt{x^2+y^2+z^2} = \begin{cases} a(\xi-\eta) & \text{with displacement} \\ a\sqrt{\xi^2+\eta^2-1} & \text{without} \end{cases}$$

Pull-back to the $\{x,y\}$-plane is accomplished by setting $\sin\phi = 0$ in (109); we then compute

$$(ds)^2 = a^2\,\tfrac{\hat\xi^2-\hat\eta^2}{\hat\xi^2-1}(d\xi)^2 + a^2\,\tfrac{\hat\xi^2-\hat\eta^2}{1\ -\hat\eta^2}(d\eta)^2 \tag{111}$$

and notice that $\{\xi,\eta\} \to \{\hat\xi,\hat\eta\}$ has caused the metric to lose its formerly conformal structure, with the consequence that the descriptioin of ∇^2 loses its former simplicity:

$$\nabla^2 = \tfrac{\sqrt{(\hat\xi^2-1)(1-\hat\eta^2)}}{a^2(\hat\xi^2-\hat\eta^2)}\Big\{\tfrac{\partial}{\partial\hat\xi}\big[\tfrac{\hat\xi^2-1}{1-\hat\eta^2}\big]^{+\frac12}\tfrac{\partial}{\partial\hat\xi} + \tfrac{\partial}{\partial\hat\eta}\big[\tfrac{\hat\xi^2-1}{1-\hat\eta^2}\big]^{-\frac12}\tfrac{\partial}{\partial\hat\eta}\Big\} \tag{112}$$

We have previously had occasion to notice that $\{r,\theta\} \rightleftarrows \{s,\theta\}$ entails both costs and benefits: the former system serves well the purposes of analysis (when expanded to include ϕ it supports the standard formulation of the theory of spherical harmonics) but only the latter serves the specialized needs of Liouville. The situation with regard to $\{\hat\xi,\hat\eta\} \rightleftarrows \{\xi,\eta\}$ is similar.

14. Spherical separation of the Schrödinger equation. Returning with (106.2) to (105), we have

$$\Big\{-\tfrac{\hbar^2}{2m}\Big[\tfrac{1}{r^2}\tfrac{\partial}{\partial r}r^2\tfrac{\partial}{\partial r} + \tfrac{1}{r^2}\big\{\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}\sin\theta\tfrac{\partial}{\partial\theta} + \tfrac{1}{\sin^2\theta}\big(\tfrac{\partial}{\partial\phi}\big)^2\big\}\Big] - \tfrac{k}{r}\Big\}\psi = E\psi$$

after simplifications. Separation proceeds from the assumption (compare (74)) that the wave function has the factored design

$$\psi(r,\theta,\phi) = R(r)\cdot\Theta(\theta)\cdot\Phi(\phi) \tag{113}$$

An initial separation gives

$$\Big\{\tfrac{1}{r^2}\tfrac{d}{dr}r^2\tfrac{d}{dr} + \tfrac{2m}{\hbar^2}\big[E+\tfrac{k}{r}\big] - \tfrac{\lambda}{r^2}\Big\}R(r) = 0 \tag{114.1}$$

$$\Big\{\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}\sin\theta\tfrac{\partial}{\partial\theta} + \tfrac{1}{\sin^2\theta}\big(\tfrac{\partial}{\partial\phi}\big)^2\Big\}Y(\theta,\phi) = -\lambda\cdot Y(\theta,\phi) \tag{114.2}$$

where λ is a spearation constant and $Y(\theta,\phi) \equiv \Theta(\theta)\cdot\Phi(\phi)$. Allusion to the Coulomb potential is confined to the "radial equation (114.1); to say the same thing another way...

The angular equation (114.2) contains no reference to the force strength k, and arises *whatever* the central potential $U(r)$. It refers to the *rotational symmetry* of such problems, and gives rise to the familiar theory of "spherical harmonics," which is but another name for the "quantum theory of angular momentum." To summarize the highlights of that well known subject, treated in every text:[41] physically acceptable solutions of (114.2) are found to arise if an only if $\lambda = \ell(\ell+1)$ with $\ell \in \{0,1,2,\dots\}$, and are found to be of the form

$$Y_\ell^m(\theta,\phi) = (\text{normalization factor})\cdot e^{im\phi}P_\ell^m(\cos\theta) \tag{115}$$

where $m \in \{-\ell,\dots,+\ell\}$ and the normalization factor is designed to achieve *orthonormality on the sphere*:

$$\int_0^{2\pi}\int_0^{\pi} \overline{Y_\ell^m(\theta,\phi)}\, Y_{\ell'}^{m'}(\theta,\phi)\sin\theta\, d\theta d\phi = \delta_{\ell\ell'}\delta_{mm'}$$

Contact with the theory of angular momentum is made explicit by the equations

$$\mathbf{L}^2\, Y_\ell^m = \ell(\ell+1)\hbar^2\cdot Y_\ell^m \quad \text{and} \quad \mathbf{L}_{\text{polar}}Y_\ell^m = m\hbar\cdot Y_\ell^m \tag{116}$$

(Here I have written $\mathbf{L}_{\text{polar}}$ where one expects to see $\mathbf{L}_z$ because, though it is conventional to identify the z-axis with the polar axis of the spherical coordinate system, I have found it more natural to assign that distinction to the x-axis; the notation $\mathbf{L}_{\text{polar}}$ is intended to capture the essence of the matter in a convention-independent way.) Finally, if $\mathbb{R}$ is a rotation matrix then

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} = \mathbb{R}\,\boldsymbol{x} \quad \text{induces} \quad Y_\ell^m \to \tilde{Y}_\ell^m = \sum_{n=-\ell}^{n=+\ell} R^m{}_n(\ell)Y_\ell^n$$

where $\|R^m{}_n(\ell)\|$ provides a $(2\ell+1)\times(2\ell+1)$-dimensional matrix representation of the action of $O(3)$.

Turning now to the radial equation (114.1)—where $\lambda = \ell(\ell+1)$ has now to be understood—one might expect, after imposition of physical side conditions, to have

$$\left\{-\tfrac{\hbar^2}{2m}\tfrac{1}{r^2}\tfrac{d}{dr}r^2\tfrac{d}{dr} - \tfrac{k}{r} + \tfrac{\hbar^2}{2m}\tfrac{\ell(\ell+1)}{r^2}\right\}R_\ell(r) = E(\ell)R_\ell(r) \tag{117}$$

with

$$\text{degeneracy of } E_n(\ell) = 2\ell+1 : \text{ number of } Y_\ell^m \text{ with the given } \ell$$

[41] I particularly like the uncluttered discussion to be found in Chapter 7 of J. Powell & B. Crassman, *Quantum Mechanics* (1961). See also my "Algebraic theory of spherical harmonics" (1996).

and that is indeed the situation typical of *arbitrary* central potentials $U(r)$. But the case $U(r) = -k/r$ is special: it emerges in that case that if we agree to write

$$\big\{E_n(\ell)\big\} \equiv \text{spectral set associated with case } \ell$$

then

$$\big\{E_n(\ell)\big\} \subset \big\{E_n(\ell-1)\big\} \subset \cdots \subset \big\{E_n(0)\big\}$$

on which basis we anticipate that

$$\begin{aligned}
&E_1 \text{ is 1-fold degenerate}\\
&E_2 \text{ is } 1+3 = 2^2\text{-fold degenerate}\\
&E_3 \text{ is } 1+3+5 = 3^2\text{-fold degenerate}\\
&\vdots\\
&E_n \text{ is } 1+3+\cdots+(2n-1) = n^2\text{-fold degenerate}
\end{aligned}$$

I have alluded here to facts—illustrated in Figure 14—usually obtained by explicit solution of the radial equation;[42] one finds that (117) possesses physically acceptable (bound state) solutions if and only if

$$\begin{aligned}
E(\ell) = E_{\nu,\ell} \equiv -E_0 \tfrac{1}{(\nu+\ell+1)^2} \quad &: \quad \nu = \ell, \ell+1, \ell+2, \ldots \\
E_0 \equiv \tfrac{mk^2}{2\hbar^2} &
\end{aligned} \tag{118}$$

and that in those cases

$$R_\ell(r) = (\text{normalization factor}) \cdot e^{-\frac{1}{2}x} x^{\frac{\alpha-1}{2}} L_\nu^\alpha(x) \tag{119}$$

where $\alpha \equiv 2\ell+1$ and $x \equiv \frac{2}{\nu+\ell+1}(r/a_0)$ with $a_0 \equiv \hbar^2/mk =$ "Bohr radius." One is motivated by the structure of these results to introduce the so-called "principal quantum number"

$$n = n(\nu,\ell) \equiv \nu + \ell + 1 \quad : \quad n = 1, 2, 3, \ldots \tag{120}$$

in terms of which one has radial eigenfunctions

$$R_{n\ell}(r) = (\text{normalization factor}) \cdot e^{-r/na_0} (2r/na_0)^\ell L_{n-\ell-1}^{2\ell+1}(2r/na_0) \tag{121}$$

and associated eigenvalues with depend upon n and ℓ not individually but only in the combination $n(\nu,\ell)$

$$E_n = -E_0 \tfrac{1}{n^2} \tag{122}$$

It is this fact—degeneracy over and above the m-independence which arises from rotational symmetry—to which the phrase "accidental degeneracy" refers. We

[42] See, for example, Schiff (Reference 59, §16) for a good account of the tedious details.

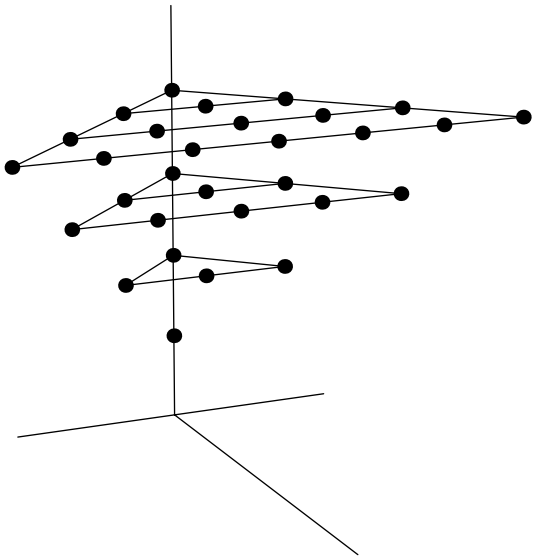

FIGURE 14: *Degeneracy of the energy spectrum in the 3-dimensional Kepler problem. The principal quantum number n runs $\uparrow$, while ℓ runs $\searrow$ and m runs $\nearrow$. Symmetry with respect to rotations in physical 3-space accounts only for the equivalence of states of a given ℓ, which fold among themselves to yield $(2\ell+1)$-dimensional respresentations of $O(3)$. All n^2 states on the n^{th} tier fold among themselves to yield representations of the "accidental symmetry" group $O(4)$, with generators $\{\mathsf{L}_x, \mathsf{L}_y, \mathsf{L}_z, \mathsf{K}_x, \mathsf{K}_y, \mathsf{K}_z\}$. $O(4)$ is here an instance of a symmetry which arises not from overt geometrical considerations, but from the collective structure of the equations of motion (not from configuration space, but from phase space); such groups are called "dynamical groups."*

read in (117) a refinement of (118), which was itself achieved already by the Bohr model (1913).

That there is yet another—and for our purposes a more informative—way to obtain (117) had been discovered by Pauli already in 1925. Pauli worked[43] in the algebraic language supplied by the then-brand-new Heisenberg formalism (the Schrödinger formalism was not published until later in 1926), and made critical use of the quantum analog of the Lenz vector $\boldsymbol{K}$. But Pauli's argument—

[43] W. Pauli, "Uber das Wasserstoffspektrum vom Staandpunkt der neuen Quantenmechanik," Z. Physik **36**, 336 (1926).

the deeper significance of which was first comprehended by V. Fock[44] and by V. Bargmann[45] after the lapse of a full decade—is fairly intricate,[46] and it is partly in an effort to discard some of the distracting detail that I propose to...

Look now, therefore, to the toy theory that arises from the pretense that "the orbital plane is all there is." Introducing (108.2) into

$$\Big\{-\tfrac{\hbar^2}{2m}\nabla^2 - k\tfrac{1}{\sqrt{x^2+y^2}}\Big\}\psi = E\psi \tag{123}$$

and assuming the wave function to possess the factored design

$$\psi(r,\theta,\phi) = R(r)\cdot Y(\theta) \tag{124}$$

we obtain

$$\Big\{-\tfrac{\hbar^2}{2m}\big[\tfrac{1}{r}\tfrac{\partial}{\partial r}r\tfrac{\partial}{\partial r} + \tfrac{1}{r^2}\big(\tfrac{\partial}{\partial\theta}\big)^2\big] - \tfrac{k}{r}\Big\}\psi = E\psi \tag{125}$$

giving

$$\Big\{\tfrac{1}{r}\tfrac{d}{dr}r\tfrac{d}{dr} + \tfrac{2m}{\hbar^2}\big[E + \tfrac{k}{r}\big] - \tfrac{\lambda}{r^2}\Big\}R(r) = 0 \tag{126.1}$$

$$\big(\tfrac{d}{d\phi}\big)^2 Y(\theta) = -\lambda\cdot Y(\theta) \tag{126.2}$$

The latter equation is k-independent (would arise whatever the design of the central potential $U(r)$) and gives

$$Y_\ell(\theta) = \tfrac{1}{\sqrt{2\pi}}e^{i\ell\theta} \quad : \quad \ell = 0, \pm1, \pm2, \ldots \tag{127}$$

which entails $\lambda = \ell^2$. The functions $Y_\ell(\theta)$ support what might be called the "theory of circular harmonics,"[47] otherwise known as the "2-dimensional quantum theory of angular momentum;" one has

$$\mathsf{L}Y_\ell(\theta) = \hbar\ell\cdot Y_\ell(\theta) \quad \text{with} \quad \mathsf{L} \equiv x\big(\tfrac{\hbar}{i}\tfrac{\partial}{\partial y}\big) - y\big(\tfrac{\hbar}{i}\tfrac{\partial}{\partial x}\big) = \tfrac{\hbar}{i}\tfrac{\partial}{\partial\theta} \tag{128}$$

Physically acceptable (bound state) solutions of the radial equation

$$\Big\{-\tfrac{\hbar^2}{2m}\tfrac{1}{r}\tfrac{d}{dr}r\tfrac{d}{dr} - \tfrac{k}{r} + \tfrac{\hbar^2}{2m}\tfrac{\ell^2}{r^2}\Big\}R_\ell(r) = E(\ell)R_\ell(r) \tag{129}$$

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[44] "Zur Theorie des Wasserstoffatoms,"Z. Physik **98**, 145 (1935). Fock gives no indication that he was aware of Pauli's earlier work.

[45] "Zur Theorie des Wasserstoffatoms: Bemerkungen zur gleichnamigen Arbeit von V. Fock," Z. Physik **99**, 576 (1936). Bargmann draws immediate attention to the fact connection between Fock's work and Pauli's, upon which he elaborates.

[46] For an excellent summary of the details see §30 in Schiff (Reference 59).

[47] See the final paragraph of §2 in Reference 1.

arise[48] if and only if

$$E(\ell) = E_{\nu,\ell} \equiv -E_0 \tfrac{1}{(\nu+|\ell|+\frac{1}{2})^2} \quad : \quad \nu = |\ell|, |\ell|+1, |\ell|+2, \ldots \tag{130}$$

and in those cases one has

$$R_\ell(r) = (\text{normalization factor}) \cdot e^{-\frac{1}{2}x} x^{\frac{\alpha}{2}} L_\nu^\alpha(x) \tag{131}$$

where now $\alpha \equiv 2|\ell|$ and $x \equiv \frac{2}{\nu+|\ell|+\frac{1}{2}}(r/a_0)$, though E_0 and a_0 have retained their former definitions. If one adopts this slight modification

$$n = n(\nu,\ell) \equiv \nu + |\ell| + 1 \quad : \quad n = 1, 2, 3, \ldots$$

of our former definition of the "principal quantum number" then we are enabled to write

$$E_n = -E_0 \tfrac{1}{(n-\frac{1}{2})^2} = -4E_0 \tfrac{1}{(2n-1)^2} \tag{132}$$

and (compare (121))

$$R_{n\ell}(r) = (\text{normalization factor}) \cdot e^{-\frac{1}{2}x} x^{|\ell|} L_{n-|\ell|-1}^{2|\ell|}(x) \tag{133}$$

where now $x = \frac{2}{n-\frac{1}{2}}(r/a_0) = \frac{4}{2n-1}(r/a_0)$.

We are in position now to take inventory of the consequences of dimensional retraction

$$\text{2-dimensions} \quad \longleftarrow \quad \text{3-dimensions}$$

The induced adjustment

$$\text{circular harmonics } Y_\ell(\theta) \quad \longleftarrow \quad \text{spherical harmonics } Y_\ell^m(\theta,\phi)$$

is profound but unproblematic. The adjustment

$$R_{n\ell}(r) \sim e^{-\frac{1}{2}x} x^{|\ell|} L_{n-|\ell|-1}^{2|\ell|}(x) \quad \longleftarrow \quad R_{n\ell}(r) \sim e^{-\frac{1}{2}x} x^{\ell} L_{n-\ell-1}^{2\ell+1}(x)$$

seems function-theoretically modest on its face, but note: it presumes

$$x = \tfrac{2}{n-\frac{1}{2}}(r/a_0) \quad \longleftarrow \quad x = \tfrac{2}{n}(r/a_0)$$

which at $n = 1$ reads $x = 4(r/a_0) \leftarrow x = 2(r/a_0)$; the radial function $R_{n\ell}(r)$ has been rendered *spatially more compact*. Which is a little surprising, for the Bohr model is effectively 2-dimensional, and it gives "orbits of the correct diameter" (by which we really mean only that it gives the observed spectroscopy). This development is consistent with

$$E_n = -E_0 \tfrac{1}{(n-\frac{1}{2})^2} \quad \longleftarrow \quad E_n = -E_0 \tfrac{1}{n^2}$$

according to which

Dimensional reduction depresses the energy spectrum

[48] Here I borrow (except for misprints) from B. Zaslow & M. E. Zandler, "Two-dimensional analog to the hydrogen atom," AJP **35**, 1118 (1967).

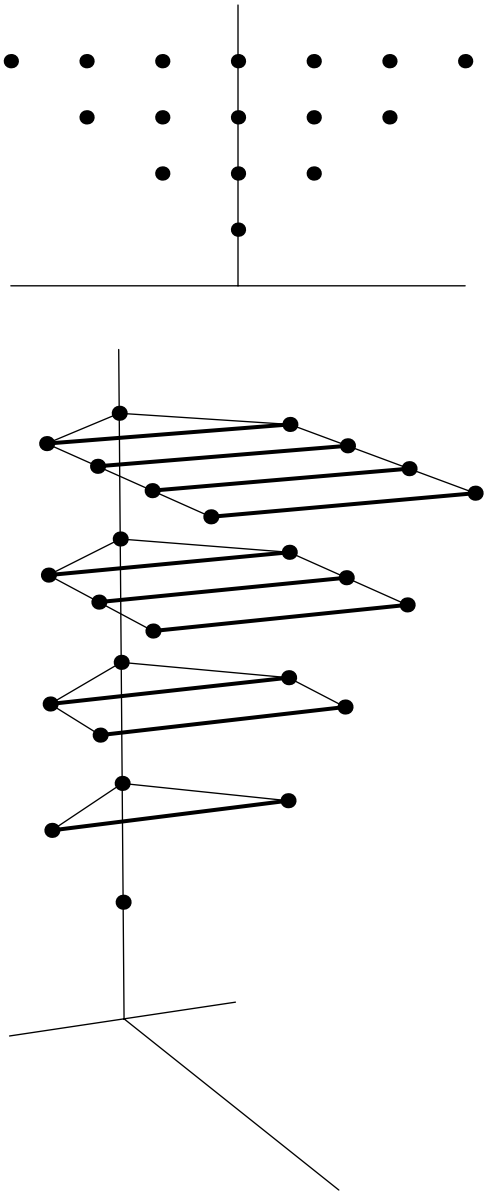

FIGURE 15: *Degeneracy of the energy spectrum in the 2-dimensional Kepler problem. In the upper display n runs $\uparrow$, ℓ runs $\leftrightarrow$. In the lower display—designed to mimic the virtues of Figure 14—n runs $\uparrow$, ℓ runs $\searrow$ and the $\swarrow\!\!\nearrow$ axis distinguishes states with respect to helicity ($+\ell$ from $-\ell$).*

Specifically, the ground state of the 2-dimensional hydrogen atom lies four times deeper than the ground state of the 3-dimensional atom. And according to R. Louden the trend continues; he reports[49] that for 1-dimensional hydrogen $E_0 = -\infty$.[50] Finally, dimensional reduction radically alters the degeneracy structure of the energy spectrum—from that shown in Figure 14 to that shown in Figure 15. More specifically, we have

$$\left.\begin{array}{l}\text{degeneracy with respect to helicity}\\ \text{(sign of } \ell\text{) is geometrical, but}\\ \\ \text{degeneracy with respect } \ell\\ \text{is accidental}\end{array}\right\} \leftarrow \left\{\begin{array}{l}\text{degeneracy with respect to } m\\ \text{is geometrical, but}\\ \\ \text{degeneracy with respect } \ell\\ \text{is accidental}\end{array}\right.$$

which puts us in position at last to consider the...

15. 2-dimensional analog of Pauli's argument. We have

$$\mathsf{H} = \tfrac{1}{2m}\big[\,\mathsf{p}_x^2 + \mathsf{p}_y^2\,\big] - k\big[\,\mathsf{x}^2 + \mathsf{y}^2\,\big]^{-\frac{1}{2}} \tag{134}$$

and at (128) acquired interest in

$$\mathsf{L} = \mathsf{x}\,\mathsf{p}_y - \mathsf{y}\,\mathsf{p}_x \tag{135}$$

With labor[51] we confirm that

$$[\,\mathsf{H}, \mathsf{L}\,] = \mathsf{O} \tag{136}$$

With Pauli (who was content to borrow from Lenz[52]) we borrow from (38) these hermitianized counterparts of K_x and K_y (K_z being of no present interest)

$$\left.\begin{array}{l}\mathsf{K}_x \equiv +\tfrac{1}{2m}\big[\,\mathsf{p}_y\mathsf{L} + \mathsf{L}\,\mathsf{p}_y\big] - k\,\mathsf{x}\big[\,\mathsf{x}^2 + \mathsf{y}^2\,\big]^{-\frac{1}{2}}\\ \mathsf{K}_y \equiv -\tfrac{1}{2m}\big[\,\mathsf{p}_x\mathsf{L} + \mathsf{L}\,\mathsf{p}_x\big] - k\,\mathsf{y}\big[\,\mathsf{x}^2 + \mathsf{y}^2\,\big]^{-\frac{1}{2}}\end{array}\right\} \tag{137}$$

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[49] "One-dimensional hydrogen atom," AJP **27**, 649 (1959).

[50] Zaslow & Zandler remark that the boxed statement pertains also to particle-in-a-box problems, and to oscillators. The question therefore arises: *Can one show that* $\boxed{}$ *holds generally?* Such a result seems intuitively plausible, but would be in one respect anomalous: for a particle in an a-by-b box one has

$$E_0 = \tfrac{\pi^2\hbar^2}{2m}\big[\tfrac{1}{a^2} + \tfrac{1}{b^2}\big] \quad \text{which} \left\{\begin{array}{l}\textit{grows} \text{ as } b \downarrow 0, \text{ but}\\ \\ \text{assumes a } \textit{reduced} \text{ value when}\\ \text{the } b\text{-dimension is discarded}\end{array}\right.$$

We touch here on this deep problem: How does one properly incorporate holonomically constrained coordinates into quantum theory?

[51] I ask *Mathematica* to confirm that $\mathsf{HL}f(x,y) = \mathsf{LH}f(x,y)$ for all $f(x,y)$, with $\mathsf{L} = \beta(x\partial_y - y\partial_x)$, etc., β being a place holder for $\hbar/i$.

[52] W. Lenz, "Über den Bewegungsverlauf und die Quantenzustände der gestörten Keplerbewegung," Z. Physik **24**, 197 (1924). Lenz cites C. Runge's

and at length establish the following commutation relations:

$$[\,\mathsf{H},\mathsf{K}_x\,]=[\,\mathsf{H},\mathsf{K}_y\,]=\mathsf{O} \tag{138}$$

$$\begin{aligned}[\,\mathsf{K}_x,\mathsf{K}_y\,]&=-i\hbar\,\tfrac{2}{m}\mathsf{H}\,\mathsf{L}\\ [\,\mathsf{K}_y,\ \mathsf{L}\ \,]&=+i\hbar\,\mathsf{K}_x\\ [\ \mathsf{L}\ ,\mathsf{K}_x\,]&=+i\hbar\,\mathsf{K}_y\end{aligned}$$

Within each eigenspace of H we can replace the operator by its eigenvalue E, which for bound states is negative. It makes sense, therefore, to introduce (compare (47.2))

$$\left.\begin{aligned}\mathsf{J}_1&\equiv\mathsf{K}_x\big/\sqrt{-\tfrac{2}{m}E}\\ \mathsf{J}_2&\equiv\mathsf{K}_y\big/\sqrt{-\tfrac{2}{m}E}\\ \mathsf{J}_3&\equiv\mathsf{L}\end{aligned}\right\} \tag{139}$$

in terms of which we have

$$\left.\begin{aligned}\big[\,\mathsf{J}_1,\mathsf{J}_2\,\big]&=i\hbar\,\mathsf{J}_3\\ \big[\,\mathsf{J}_2,\mathsf{J}_3\,\big]&=i\hbar\,\mathsf{J}_1\\ \big[\,\mathsf{J}_3,\mathsf{J}_1\,\big]&=i\hbar\,\mathsf{J}_2\end{aligned}\right\} \tag{140}$$

and therefore (immediately) $\big[\,\mathsf{J}^2,\mathsf{J}_1\,\big]=\big[\,\mathsf{J}^2,\mathsf{J}_2\,\big]=\big[\,\mathsf{J}^2,\mathsf{J}_3\,\big]=\mathsf{O}$ with

$$\begin{aligned}\mathsf{J}^2&\equiv\mathsf{J}_1^2+\mathsf{J}_2^2+\mathsf{J}_3^2\\ &=\mathsf{L}^2-\tfrac{m}{2E}\mathsf{K}^2\end{aligned} \tag{141}$$

Classically

$$\begin{aligned}K^2\equiv K_x^2+K_y^2&=\big[\tfrac{1}{m}p_y(xp_y-yp_x)-\tfrac{k}{r}x\big]^2+\big[\tfrac{1}{m}p_x(yp_x-xp_y)-\tfrac{k}{r}y\big]^2\\ &=\tfrac{2}{m}\big[\tfrac{1}{2m}(p_x^2+p_y^2)-\tfrac{k}{r}\big]\big[xp_y-yp_x\big]^2+k^2\\ &=\tfrac{2}{m}HL^2+k^2\end{aligned}$$

so we are not surprised to discover by (heavy!) exploratory calculation that

$$\mathsf{K}^2\equiv\mathsf{K}_x^2+\mathsf{K}_y^2=\tfrac{2}{m}\mathsf{H}\,(\,\mathsf{L}^2+\tfrac{1}{4}\hbar^2\mathsf{I}\,)+k^2\,\mathsf{I} \tag{142}$$

Retreating now again to the (bound state) eigenspace $\mathcal{H}_E\subset\mathcal{H}$, we have

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[continued from the preceding page] *Vektoranalysis* (1919) as his own source, and, according to Goldstein, Runge borrowed his material from Gibbs (∼1900). The *Dictionary of Scientific Biography* makes no mention of Lenz, but provides a fascinating account of the life of Carl Runge (1856–1927), who appears to have been introduced to his future wife, to have been motivated to move from pure mathematics into spectral physics, and ultimately to have become the father-in-law of Richard Courant—all in consequence of having been tall, handsome, and an outstanding ice skater.

$$\begin{aligned}\mathsf{J}^2 &= \mathsf{L}^2 - \tfrac{m}{2E}\Big\{\tfrac{2}{m}E\,(\,\mathsf{L}^2 + \tfrac{1}{4}\hbar^2\mathsf{I}\,) + k^2\,\mathsf{I}\Big\} \\ &= -\big[\tfrac{1}{4}\hbar^2 + \tfrac{mk^2}{2E}\big]\mathsf{I} \end{aligned} \tag{143}$$

Borrowing from the algebraic formulation of the 3-dimensional quantum theory of angular momentum[53] we know that

$$\mathsf{J}^2 \text{ has eigenvalues } j(j+1)\hbar^2 \text{ with } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

and that

$$\text{the eigenvalue } j(j+1)\hbar^2 \text{ is } (2j+1)\text{-fold degenerate}$$

Bring this information to (143) we obtain

$$\begin{aligned} -\tfrac{mk^2}{2E} &= \big[j(j+1) + \tfrac{1}{4}\big]\hbar^2 \\ &= \tfrac{1}{4}(2j+1)^2\hbar^2 \end{aligned}$$

giving

$$\begin{aligned} E(j) &= -4\tfrac{mk^2}{2\hbar^2}\tfrac{1}{(2j+1)^2} \\ &= -4E_0\tfrac{1}{(\text{integer})^2} \end{aligned} \tag{144}$$

Whether one argues analytically (with Schrödinger) or group-theoretically (with Pauli), one has Bohr's result

$$E = -E_0\tfrac{1}{(\text{integer})^2} \qquad \text{in the 3-dimensional case}$$

while at (132) Zaslow & Zandler's analytical argument yielded

$$E = -4E_0\tfrac{1}{(\text{odd integer})^2} \quad \text{in the 2-dimensional case}$$

with the implication that at (144) we must *exclude the fractional values of* j. With respect to description of the "hidden symmetry" of the 2-dimensional hydrogen atom one must disallow the spinor representations of the rotation group; the "dynamical group" of that system is (we speak only of the bound states) $O(3)$, not $SU(2)$.[54]

It is, in this light, more natural to assign to j the work formerly assigned to the principal quantum number n; the identification $2j+1 = 2n-1$ entails

$$n = j+1 \quad : \quad j = 0, 1, 2, \ldots$$

so we acquire the notations

$$E_n = E_{j+1} = E(j)$$

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[53] See, for example, Schiff's §27, Mertzbacher's Chapter 16, or §§4.3 & 4.4 in D. Griffiths' *Introduction to Quantum Mechanics* (1995).

[54] It is now easy to understand how confusion on this point might arise, as historically it did arise. For detailed discussion and references see A. Cisneros & H. V. McIntosh, "Symmetry of the two-dimensional hydrogen atom," J. Math. Phys. **10**, 277 (1969). I return to this problem on p. 77 below.

Turning now again to description (within the 2-dimensional theory) of the hydrogenic wavefunctions (i.e., to construction of a natural basis in $\mathcal{H}_{E(j)}$) we again borrow from the 3-dimensional quantum theory of angular momentum, introducing

$$\left.\begin{aligned} \mathsf{J}_+ &\equiv \mathsf{J}_1 + i\mathsf{J}_2 = (\,\mathsf{K}_x + i\mathsf{K}_y)\sqrt{-m/2E(j)} \\ \mathsf{J}_- &\equiv \mathsf{J}_1 - i\mathsf{J}_2 = (\,\mathsf{K}_x - i\mathsf{K}_y)\sqrt{-m/2E(j)} \end{aligned}\right\} \tag{145}$$

From (140) it then follows that

$$\left.\begin{aligned} \big[\,\mathsf{J}^2, \mathsf{J}_\pm\,\big] &= \quad \mathsf{O} \\ \big[\,\mathsf{L}, \mathsf{J}_\pm\,\big] = \big[\,\mathsf{J}_3, \mathsf{J}_\pm\,\big] &= \pm\hbar\,\mathsf{J}_\pm \end{aligned}\right\} \tag{146}$$

The (non-hermitian) "ladder operators" $_\pm$ are introduced because—as is established by an algebraic argument which it must have given immeasurable satisfaction to its inventor, but which it would be pointless to reproduce—they permit one to climb up and down through the set

$$|j,\ell) \quad : \quad \ell = -j, \ldots, -1, 0, +1, \ldots, +j$$

of simultaneous eigenstates of J^2 and L:

$$\mathsf{J}^2|j,\ell) = j(j+1)\hbar^2\,|j,\ell) \qquad \text{and} \qquad \mathsf{L}|j,\ell) = \ell\hbar\,|j,\ell)$$

In 3-dimensional physics one encounters operators $\{\mathsf{L}_x, \mathsf{L}_y, \mathsf{L}_z\}$ which are (as we have had occasion already to remark) algebraically identical to the operators $\{\mathsf{J}_x, \mathsf{J}_y, \mathsf{J}_z\}$ of present interest. In that context the operators $\{\mathsf{L}_\pm, \mathsf{L}_z, \mathsf{L}^2\}$ acquire representations

$$\begin{aligned} \mathsf{L}_\pm &= \pm\hbar e^{\pm i\phi}\big(\tfrac{\partial}{\partial\theta} \pm i\cot\theta\tfrac{\partial}{\partial\phi}\big) \\ \mathsf{L}_z &= \tfrac{\hbar}{i}\tfrac{\partial}{\partial\phi} \\ \mathsf{L}^2 &= \mathsf{L}_+\mathsf{L}_- + \mathsf{L}_z^2 - \hbar\mathsf{L}_z \end{aligned}$$

and the eigenstates $|\ell, m)$ become just the familiar spherical harmonics $Y_\ell^m(\theta,\phi)$. The 2-dimensional Kepler problem gives rise, however, to quite a different realization of the same algebra. Introducing

$$\left.\begin{aligned} \mathsf{x} &= r\cos\theta \\ \mathsf{y} &= r\,\sin\theta \\ \mathsf{p}_x &= \tfrac{\hbar}{i}\big\{\,\cos\theta\cdot\tfrac{\partial}{\partial r} - \tfrac{1}{r}\sin\theta\cdot\tfrac{\partial}{\partial\theta}\big\} \\ \mathsf{p}_y &= \tfrac{\hbar}{i}\big\{\,\sin\theta\cdot\tfrac{\partial}{\partial r} + \tfrac{1}{r}\cos\theta\cdot\tfrac{\partial}{\partial\theta}\big\} \end{aligned}\right\} \tag{147}$$

into first (135) and then (137) we obtain

$$\left.\begin{aligned} \mathsf{L} &= \tfrac{\hbar}{i}\tfrac{\partial}{\partial\theta} \\ \mathsf{K}_+ &= e^{i\theta}\big[\tfrac{\hbar^2}{2mr}\big(1 - 2i\tfrac{\partial}{\partial\theta}\big)\big(r\tfrac{\partial}{\partial r} + i\tfrac{\partial}{\partial\theta}\big) + k\big] \\ \mathsf{K}_- &= \text{complex conjugate of the above} \end{aligned}\right\} \tag{148}$$

and confirm by explicit calculaton that the operators thus described to in fact satisfy the anticipated commutation relations. With (148) in hand, we are in position to construct polar representations of $\mathbf{J}^2$ and $\mathbf{J}_\pm$, though in this regard two interrelated points should be borne in mind:

- Within the j^{th} energy eigenspace $\mathcal{H}_{E(j)}$ the action of $\mathbf{J}^2$ is almost trivial: if $|\psi) \in \mathcal{H}_{E(j)}$ then $\mathbf{J}^2|\psi) = j(j+1)\hbar^2|\psi)$;
- The presence of the factor $\sqrt{-m/2E(j)} = \frac{\hbar}{2k}(2j+1)$ in the definition (145) of the ladder operators $\mathbf{J}_\pm$ gives those operators a j-dependence (their design is specific to each separate $\mathcal{H}_{E(j)}$) which has no counterpart in the theory of angular momentum (i.e., in the design of $\mathbf{L}_\pm$).

With the aid of

$$\mathbf{J}_\pm = \tfrac{\hbar}{2k}(2j+1)\cdot\mathbf{K}_\pm \tag{149.1}$$

$$= \tfrac{2j+1}{2}\hbar \begin{cases} e^{i\theta}\big[a_0\tfrac{1}{2r}\big(1-2i\tfrac{\partial}{\partial\theta}\big)\big(r\tfrac{\partial}{\partial r}+i\tfrac{\partial}{\partial\theta}\big)+1\big] \\ \text{complex conjugate of the above} \end{cases} \tag{149.2}$$

one can hike right/left on the j^{th} tier of the top design in Figure 15, or clamber about on the j^{th} branch of the tree shown at bottom. I illustrate by example how this works:

In the discussion culminating in (133) we obtained separated eigenfunctions which (by $n = j+1$) can be described

$$\Psi_{j\ell}(r,\theta) = (\text{normalization factor})\cdot F_{j,\ell}(r,\theta)$$
$$F_{j,\ell}(r,\theta) \equiv e^{-\frac{1}{2}x}x^{|\ell|}L^{2|\ell|}_{j-|\ell|}(x)\cdot e^{i\ell\theta}$$

with $x \equiv \frac{4}{2j+1}(r/a_0)$. It follows, in particular, that

$$F_{2,0}(r,\theta) = e^{-\frac{1}{2}x}L^0_2(x) \quad \text{with} \quad x = \tfrac{4}{5a_0}r$$

Mathematica, by computation based upon (149.2), gives

$$\begin{aligned}\mathbf{J}_+F_{2,0}(r,\theta) &= -\hbar\,\tfrac{4}{25a_0^2}\exp\big\{-\tfrac{1}{2}\tfrac{4}{5a_0}r\big\}(15a_0-4r)re^{i\theta}\\ &= -\hbar\,e^{-\frac{1}{2}x}xL^2_1(x)e^{i\theta}\\ &= -\hbar\,F_{2,1}(r,\theta)\end{aligned}$$

$$\begin{aligned}\mathbf{J}_+F_{2,1}(r,\theta) &= -\hbar\,\tfrac{16}{25a_0^2}\exp\big\{-\tfrac{1}{2}\tfrac{4}{5a_0}r\big\}r^2e^{2i\theta}\\ &= -\hbar\,e^{-\frac{1}{2}x}x^2L^4_0(x)e^{2i\theta}\\ &= -\hbar\,F_{2,2}(r,\theta)\end{aligned}$$

$$\mathbf{J}_+F_{2,2}(r,\theta) = 0$$

These results, since they conform precisely to our expectations, inspire confidence in the accuracy of (149.2) (and confidence also that I have correctly adjusted many mistaken signs in the paper by Zaslow & Zandler[48] upon which I have relied).

We will have occasion to revisit Pauli's lovely argument. But for the moment I am content simply to emphasize that it does go through in the 2-dimensional case, and—compare Schiff's §30—works more simply than in the 3-dimensional case, upon which the 2-dimensional argument sheds useful light.[55] Initially (note the subscripts that appear in (135) and (137)) we found it convenient to embrace Cartesian tyranny, but by the end of the discussion we had regained our polar poise. Note finally that it was by a *consistency argument* that we able at (144) to obtain—unanticipatedly?—a description of the (bound state) energy spectrum. And that I have passed over in silence all that Pauli's argument has to say about unbound solutions of the 2-dimensional Kepler problem.

16. Alternate spherical separation of the Schrödinger equation. Returning with (106.3) to (105), we have an equation which can be written

$$\Big\{-\tfrac{\hbar^2}{2m}\Big[\tfrac{1}{a^2e^{3s}\sin\theta}\big\{\tfrac{\partial}{\partial s}e^s\sin\theta\tfrac{\partial}{\partial s}+\tfrac{\partial}{\partial\theta}e^s\sin\theta\tfrac{\partial}{\partial\theta}\big\} + \tfrac{1}{(ae^s\sin\theta)^2}\big(\tfrac{\partial}{\partial\phi}\big)^2\Big]-\tfrac{k}{ae^s}\Big\}\psi = E\psi \tag{150}$$

after simplifications, but which would lose its fragile symmetry if one were to further simplify the inner $\big\{\text{etc.}\big\}$ term. The variable θ is now more naturally associated with s than with ϕ, and if we assume the wave function to possess the design

$$\psi(s,\theta,\phi) = Z(s,\theta)\cdot\Phi(\phi) \tag{151}$$

we are led by familiar steps to

$$\Phi_m(\phi) = \tfrac{1}{\sqrt{2\pi}}e^{im\phi} \quad : \quad m = 0, \pm1, \pm2, \ldots \tag{152}$$

and to[56]

$$\Big\{-\tfrac{\hbar^2}{2\mathsf{M}}\Big[\tfrac{1}{a^2e^{3s}\sin\theta}\big\{\tfrac{\partial}{\partial s}e^s\sin\theta\tfrac{\partial}{\partial s}+\tfrac{\partial}{\partial\theta}e^s\sin\theta\tfrac{\partial}{\partial\theta}\big\}\Big] - \tfrac{k}{ae^s}+\tfrac{\hbar^2}{2\mathsf{M}}\tfrac{m^2}{(ae^s\sin\theta)^2}\Big\}Z_m(s,\theta) = E(m)Z_m(s,\theta)$$

where, since m has been preempted, I have been obliged to use μ to denote the mass of the particle. The latter equation is—its symmetry notwithstanding—analytically unappetizing, but if one uses

$$ae^s = r \quad \text{whence} \quad \tfrac{\partial}{\partial s} = r\tfrac{\partial}{\partial r} \tag{153}$$

it gives back

$$\Big\{-\tfrac{\hbar^2}{2\mathsf{M}}\Big[\tfrac{1}{r^2\sin\theta}\big\{\tfrac{\partial}{\partial r}r^2\sin\theta\tfrac{\partial}{\partial r}+\tfrac{\partial}{\partial\theta}\sin\theta\tfrac{\partial}{\partial\theta}\big\}\Big] - \tfrac{k}{r}+\tfrac{\hbar^2}{2\mathsf{M}}\tfrac{m^2}{r^2\sin^2\theta}\Big\}\mathcal{Z}_m(r,\theta) = E(m)\mathcal{Z}_m(r,\theta)$$

[55] Does Pauli's argument have things to say also in the n-dimensional case?

[56] Here and henceforth: whenever m acquires other work to do, I will use M to denote mass.

which when written

$$\left\{-\tfrac{\hbar^2}{2\mathsf{M}}\left[\tfrac{1}{r^2}\tfrac{\partial}{\partial r}r^2\tfrac{\partial}{\partial r}+\tfrac{1}{r^2}\left\{\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}\sin\theta\tfrac{\partial}{\partial\theta}-\tfrac{m^2}{\sin^2\theta}\right\}\right]-\tfrac{k}{r}\right\}\mathcal{Z}=E\mathcal{Z}$$

is seen to give back a familiar pair of equations when separated.

The analogous 2-dimensional theory is prettier: introducing (108.3) into (123) we obtain

$$\left\{-\tfrac{\hbar^2}{2m}\left[\tfrac{1}{a^2e^{2s}}\left\{\left(\tfrac{\partial}{\partial s}\right)^2+\left(\tfrac{\partial}{\partial\theta}\right)^2\right\}\right]-\tfrac{k}{ae^s}\right\}\psi(s,\theta)=E\psi(s,\theta) \tag{154}$$

which separates straightforwardly into a pair of equations

$$\left.\begin{aligned}\left\{\tfrac{1}{a^2e^{2s}}\left(\tfrac{d}{ds}\right)^2+\tfrac{2m}{\hbar^2}\left[E+\tfrac{k}{ae^s}\right]-\tfrac{\ell^2}{a^2e^{2s}}\right\}S(s)&=0\\ \left(\tfrac{d}{d\theta}\right)^2Y(\theta)&=-\ell^2Y(\theta)\end{aligned}\right\} \tag{155}$$

which are readily seen to be equivalent (by (153)) to (126), but which in themselves appear to offer no distinct advantages.[57] The Hamilton-Jacobi precursors of (154) were encountered at (84).

17. Quantum mechanics of Liouville systems. I digress, partly to make clear why it is that (154) is relatively simpler than its 3-dimensional counterpart, and partly to prepare for developments that lie ahead.

Liouville's theory works in n-dimensions. It will serve my purposes to look only to the cases $n=3$ (which is typical of the general case) and $n=2$, which is, as will emerge, special. Liouville, on the presumption that

$$\text{Lagrangian} = \text{kinetic energy} - \text{potential}$$

imposes strong restrictions upon the design of the kinetic energy term, and separate (but related) restrictions upon the design of the potential energy function. Looking first to the former: it was remarked already in §12 that Liouville assumes the metric to be *conformally Euclidean*

$$\|g_{ij}(q^1,q^2,q^3)\| = u(q^1,q^2,q^3)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

but with a vengence: he asssumes

$$u(q^1,q^2,q^3)=u_1(q^1)+u_2(q^2)+u_3(q^3)$$

[57] Note, however, that the system (155) does lend itself beautifully well to study of central force problems of the non-Keplerean design

$$U(x,y)=\tfrac{k}{x^2+y^2}\sim r^{-2}$$

which by the general formula which gave (106) gives

$$\nabla^2 = \begin{cases} u^{-\frac{3}{2}}\big\{\frac{\partial}{\partial q^1}\sqrt{u}\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\sqrt{u}\frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}\sqrt{u}\frac{\partial}{\partial q^3}\big\} & : \text{ 3-dimensions} \\ u^{-1}\big\{\big(\frac{\partial}{\partial q^1}\big)^2 + \big(\frac{\partial}{\partial q^2}\big)^2\big\} & : \text{ 2-dimensions} \end{cases}$$

This accounts already for the simplicity of (108.3/4/5) relative to (106.3/4/5), and for some of the simplicity of such 2-dimensional Schrödinger equations as may make use of the former ∇^2 operators.

But Liouville assumes also that the potential is of the form

$$U = \tfrac{1}{u}\big\{w_1(q^1) + w_2(q^2) + w_3(q^3)\big\}$$

One is led, therefore, to a Schrödinger equation of a form

$$\tfrac{1}{u}\sum_{i=1}^{3}\Big\{-\tfrac{\hbar^2}{2m}\big[\big(\tfrac{\partial}{\partial q^i}\big)^2 + \tfrac{1}{2u}u_i'\tfrac{\partial}{\partial q^i}\big] + w_i\Big\}\psi = E\psi$$

which can be written

$$\sum_{i=1}^{3}\Big\{-\tfrac{\hbar^2}{2m}\big[\big(\tfrac{\partial}{\partial q^i}\big)^2 + \tfrac{1}{2u}u_i'\cdot\tfrac{\partial}{\partial q^i}\big] + \big[w_i - E\,u_i\big]\Big\}\psi = 0$$

$\frac{1}{2u}u_i'$-factor messes up separability

Separability is destroyed by the presence of factors derived from the "interstitial $\sqrt{u}$-factors," which in the n-dimensional case become "interstitial $u^{\frac{n-2}{2}}$-factors;" only in the case $n = 2$ are such factors absent,[58] and in that case we have

$$\sum_{i=1}^{2}\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial q^i}\big)^2 + \big[w_i - E\,u_i\big]\Big\}\psi = 0$$

which separates instantly to give (compare the Hamilton-Jacobi system (96))

$$\left.\begin{aligned} \Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial q^1}\big)^2 + \big[w_1 - E\,u_1\big] - \epsilon_1\Big\}\psi = 0 \\ \Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial q^2}\big)^2 + \big[w_2 - E\,u_2\big] - \epsilon_2\Big\}\psi = 0 \end{aligned}\right\} \tag{156}$$

with $\epsilon_1 + \epsilon_2 = 0$.

Consider again, by way of illustration, the 2-dimensional Kepler problem in alternate polar coordinates: taking our descriptions of $u_1(s)$, $u_2(\theta)$, $w_1(s)$ and $w_2(\theta)$ from (20), we have

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial s}\big)^2 + \big[-kae^s - E\,a^2e^{2s}\big] + \epsilon\Big\}\psi = 0$$

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial \theta}\big)^2 - \epsilon\Big\}\psi = 0$$

A single-valuedness condition enforces $2m\epsilon/\hbar^2 = \ell^2$ ($\ell = 0, \pm1, \pm2, \ldots$), so we have recovered precisely (155).

[58] I dismiss as uninteresting the n-dimensional cases in which u is constant.

18. Parabolic separation of the Schrödinger equation. Returning with (106.4) to (105), we have

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{1}{(\mu^2+\nu^2)}\left\{\frac{1}{\mu}\frac{\partial}{\partial\mu}\mu\frac{\partial}{\partial\mu}+\frac{1}{\nu}\frac{\partial}{\partial\nu}\nu\frac{\partial}{\partial\nu}\right\}+\frac{1}{\mu^2\nu^2}\left(\frac{\partial}{\partial\phi}\right)^2\right]-\frac{2k}{\mu^2+\nu^2}\right\}\psi=E\psi \tag{157}$$

after elementary simplifications. Authors of most of the better quantum texts[59] —perpetuating a tradition inaugurated by Schrödinger himself—give some space to discussion of the "parabolic separability of the hydrogen problem." Most authors mention that parabolic coordinates are particularly well-adapted to discussion of the Coulomb scattering problem, and to the perturbation theory of orbitals when "a particular direction in space is distinguished by some external force [as, for example, in the] Stark effect, photo-electric effect, Compton effect, collision of electrons."[60] But (with the exception only of Cisneros & McIntosh[54]) all—including myself, on a former occasion[61]—proceed from definitions which depart from the convention established at (85); in place of

$$\begin{aligned} x &= \tfrac{1}{2}(\mu^2-\nu^2)\\ y &= \mu\nu\cos\phi\\ z &= \mu\nu\sin\phi \end{aligned}$$

they write (but without the hats)

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\hat\mu-\hat\nu)\\ y &= \sqrt{\hat\mu\hat\nu}\cos\phi\\ z &= \sqrt{\hat\mu\hat\nu}\sin\phi \end{aligned}\right\} \tag{158}$$

which one might consider to be recommended already on dimensional grounds. By computation

$$(ds)^2=\tfrac{\hat\mu+\hat\nu}{4\hat\mu}(d\hat\mu)^2+\tfrac{\hat\mu+\hat\nu}{4\hat\nu}(d\hat\nu)^2+\hat\mu\hat\nu(d\phi)^2$$

from which it follows that

$$\nabla^2=\tfrac{4}{\hat\mu+\hat\nu}\left\{\tfrac{\partial}{\partial\hat\mu}\hat\mu\tfrac{\partial}{\partial\hat\mu}+\tfrac{\partial}{\partial\hat\nu}\hat\nu\tfrac{\partial}{\partial\hat\nu}+\tfrac{\partial}{\partial\phi}\tfrac{\hat\mu+\hat\nu}{4\hat\mu\hat\nu}\tfrac{\partial}{\partial\phi}\right\} \tag{159}$$

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[59] See, for example,
D. Bohm. *Quantum Theory* (1951),§58;
L. D. Landau & E. M. Lifshitz, *Quantum Mechanics* (1958), §37;
L. I. Schiff, *Quantum Mechanics* (3rd edition 1968), pp. 95–98;
E. Merzbacher, *Quantum Mechanics* (2nd edition 1970), pp. 245–250;
or H. A. Betha & E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (1957), §6.

[60] Here I quote Bethe & Salpeter.

[61] See QUANTUM MECHANICS (1967), Chapter 2, pp. 30-33.

while pull-back to the $\{x, y\}$-plane gives

$$(ds)^2 = \tfrac{\hat{\mu}+\hat{\nu}}{4\hat{\mu}}(d\hat{\mu})^2 + \tfrac{\hat{\mu}+\hat{\nu}}{4\hat{\nu}}(d\hat{\nu})^2$$

whence

$$\nabla^2 = 4\tfrac{\sqrt{\hat{\mu}\hat{\nu}}}{\hat{\mu}+\hat{\nu}}\Big\{\tfrac{\partial}{\partial\hat{\mu}}\big[\tfrac{\hat{\mu}}{\hat{\nu}}\big]^{+\frac{1}{2}}\tfrac{\partial}{\partial\hat{\mu}} + \tfrac{\partial}{\partial\hat{\nu}}\big[\tfrac{\hat{\mu}}{\hat{\nu}}\big]^{-\frac{1}{2}}\tfrac{\partial}{\partial\hat{\nu}}\Big\} \tag{160}$$

But whether we work in three dimensions or two, we have

$$r = \tfrac{1}{2}(\hat{\mu} + \hat{\nu})$$

The pattern of events is familiar, in that $\{\hat{\mu}, \hat{\nu}\} \leftrightarrows \{\mu, \nu\}$ entails trade-offs: the former system may simplify analytical work, but only the latter system leads to the conformally Euclidean metric upon which Liouville separation depends.[62]

Looking first, in bare outline, to the parabolic solution of the 3-dimensional Kepler problem, we—non-standardly—take (157) as our point of departure, and after an initial separation have

$$\begin{aligned}\Big\{-\tfrac{\hbar^2}{2\mathsf{M}}\Big[\tfrac{1}{(\mu^2+\nu^2)}\big\{\tfrac{1}{\mu}\tfrac{\partial}{\partial\mu}\mu\tfrac{\partial}{\partial\mu} + \tfrac{1}{\nu}\tfrac{\partial}{\partial\nu}\nu\tfrac{\partial}{\partial\nu}\big\} - \tfrac{m^2}{\mu^2\nu^2}\Big] \\ - \tfrac{2k}{\mu^2+\nu^2}\Big\}P(\mu,\nu) = EP(\mu,\nu) \quad (161.1)\\ \big(\tfrac{d}{d\phi}\big)^2\Phi(\phi) = -m^2\Phi(\phi)\end{aligned}$$

By coordinate adjustment

$$\left.\begin{array}{l}\mu = \sqrt{\hat{\mu}}\\ \nu = \sqrt{\hat{\nu}}\end{array}\right\} \quad \text{whence} \quad \left\{\begin{array}{l}\frac{\partial}{\partial\mu} = 2\sqrt{\hat{\mu}}\frac{\partial}{\partial\hat{\mu}}\\ \frac{\partial}{\partial\nu} = 2\sqrt{\hat{\nu}}\frac{\partial}{\partial\hat{\nu}}\end{array}\right.$$

(160.1) assumes the form

$$\begin{aligned}\Big\{-\tfrac{\hbar^2}{2\mathsf{M}}\Big[\tfrac{4}{(\hat{\mu}+\hat{\nu})}\big\{\tfrac{\partial}{\partial\hat{\mu}}\hat{\mu}\tfrac{\partial}{\partial\hat{\mu}} + \tfrac{\partial}{\partial\hat{\nu}}\hat{\nu}\tfrac{\partial}{\partial\hat{\nu}}\big\} - \tfrac{m^2}{\hat{\mu}\hat{\nu}}\Big] \\ - \tfrac{2k}{\hat{\mu}+\hat{\nu}}\Big\}\hat{P}(\hat{\mu},\hat{\nu}) = E\hat{P}(\hat{\mu},\hat{\nu}) \qquad (161.2)\end{aligned}$$

standardly encountered in the literature. A final separation gives

$$\begin{aligned}\Big\{\tfrac{d}{d\hat{\mu}}\hat{\mu}\tfrac{d}{d\hat{\mu}} - \tfrac{m^2}{4\hat{\mu}} + \tfrac{\mathsf{M}}{2\hbar^2}E\hat{\mu} + \tfrac{\mathsf{M}}{2\hbar^2}k_1 - \epsilon_1\Big\}\hat{M}(\hat{\mu}) = 0\\ \Big\{\tfrac{d}{d\hat{\nu}}\hat{\nu}\tfrac{d}{d\hat{\nu}} - \tfrac{m^2}{4\hat{\nu}} + \tfrac{\mathsf{M}}{2\hbar^2}E\hat{\nu} + \tfrac{\mathsf{M}}{2\hbar^2}k_2 - \epsilon_2\Big\}\hat{N}(\hat{\nu}) = 0\end{aligned}$$

[62] As Whittaker remarks, if Liouville were confronted with a kinetic energy term of the form $T = \tfrac{1}{4}(\hat{\mu}+\hat{\nu})\big[\tfrac{1}{\hat{\mu}}\big(\tfrac{d\hat{\mu}}{dt}\big)^2 + \tfrac{1}{\hat{\nu}}\big(\tfrac{d\hat{\nu}}{dt}\big)^2\big]$ his first act would be to make a coordinate adjustment $\hat{\mu} \to \mu = \sqrt{\hat{\mu}}$, $\hat{\nu} \to \nu = \sqrt{\hat{\mu}}$ designed to *achieve* conformality: $T \to (\mu^2+\nu^2)\big[\dot{\mu}^2 + \dot{\nu}^2\big]$.

where $\epsilon_1 + \epsilon_2 = 0$ and $k_1 + k_2 = 2k$. In the interest of clarity we (with Schiff) select the options $\epsilon_2 = -(\epsilon_1 \equiv \epsilon)$, $k_2 = 0$ and obtain

$$\left\{ \tfrac{d}{d\hat{\mu}} \hat{\mu} \tfrac{d}{d\hat{\mu}} - \tfrac{m^2}{4\hat{\mu}} + \tfrac{\mathsf{M}}{2\hbar^2} E\hat{\mu} + \tfrac{\mathsf{M}}{\hbar^2} k - \epsilon \right\} \hat{M}(\hat{\mu}) = 0$$
$$\left\{ \tfrac{d}{d\hat{\nu}} \hat{\nu} \tfrac{d}{d\hat{\nu}} - \tfrac{m^2}{4\hat{\nu}} + \tfrac{\mathsf{M}}{2\hbar^2} E\hat{\nu} \qquad + \epsilon \right\} \hat{N}(\hat{\nu}) = 0$$

Tedious analysis motivates the introduction of the parameter

$$\alpha \equiv \sqrt{-2\mathsf{M}E/\hbar^2} \quad : \quad \text{dimensionality of (length)}^{-1}$$

and the definitions

$$\lambda_1 \equiv \tfrac{1}{\alpha}\left(\tfrac{\mathsf{M}k}{\hbar^2} - \epsilon\right)$$
$$\lambda_2 \equiv \tfrac{1}{\alpha}\epsilon$$

Physically acceptable solutions arise if and only if

$$\left.\begin{aligned} \lambda_1 &= n_1 + \tfrac{1}{2}(|m|+1) \\ \lambda_2 &= n_2 + \tfrac{1}{2}(|m|+1) \end{aligned}\right\} \quad : \quad n_1, n_2 = 0, 1, 2, \ldots$$

which give

$$\begin{aligned} E &= -\tfrac{\hbar^2\alpha^2}{2\mathsf{M}} \\ &= -\tfrac{\mathsf{M}k^2}{2\hbar^2}\tfrac{1}{n^2} \quad \text{with} \quad n \equiv \lambda_1 + \lambda_2 = (n_1 + n_2 + 1) + |m| \end{aligned} \tag{162}$$

and are in those cases (apart from normalization) given by

$$\begin{aligned} \hat{M}(\hat{\mu}) &= e^{-\frac{1}{2}\hat{\mu}} \hat{\mu}^{\frac{1}{2}|m|} L^{|m|}_{n_1+|m|}(\alpha\hat{\mu}) \\ \hat{N}(\hat{\nu}) &= \text{similar} \end{aligned} \tag{163}$$

—all of which has been known for a very long time,[63] and is of immediate interest only as counterpoint for what now follows.

To obtain a parabolic theory of the 2-dimensional hydrogen atom one might introduce (108.4) into (123). Alternatively and more efficiently, one might

[63] Schiff cites Schrödinger, "Quantisierung als Eigenwertproblem. III," Ann. Physik **80**, 437 (1926); P. S. Epstein, "The Stark effect from the point of view of Schroedinger's theory," Phys. Rev. **28**, 695 (1926) and I. Waller, "Der Starkeffekt zweiter Ordnung bei Wasserstoff und die Rydbergkorrektion der Spektra von He und Li^+," Z. Physik **38**, 635 (1926). Epstein had used parabolic coordinates to study (in language of the "Old Quantum Theory") the Stark effect already in 1916. He was at Caltech, and his paper marks, I believe, the first occurance of the Schrödinger equation in the pages of Physical Review.

Note, by the way, that we have been led to solutions which are axially (rather than rotationally) symmetric, and that the familiar spherical harmonic factor has been "disolved;" Bethe & Salpeter elaborate upon the point.

particularize (156). Both methods lead almost instantly to separated equations

$$\begin{aligned}\left\{-\tfrac{\hbar^2}{2m}\left(\tfrac{d}{d\mu}\right)^2 - E\,\mu^2 - k_1 - \epsilon_1\right\}M(\mu) = 0\\ \left\{-\tfrac{\hbar^2}{2m}\left(\tfrac{d}{d\nu}\right)^2 - E\,\nu^2 - k_2 - \epsilon_2\right\}N(\nu) = 0\end{aligned}$$

in which $k_1+k_2 = 2k$, $\epsilon_1+\epsilon_2 = 0$. These equations—which should be compared to their Hamilton-Jacobi percursors (89)—can be written

$$\left.\begin{aligned}\left\{-\tfrac{\hbar^2}{2m}\left(\tfrac{d}{d\mu}\right)^2 + \tfrac{1}{2}m\omega^2\mu^2\right\}M(\mu) = (k_1+\epsilon_1)M(\mu)\\ \left\{-\tfrac{\hbar^2}{2m}\left(\tfrac{d}{d\nu}\right)^2 + \tfrac{1}{2}m\omega^2\nu^2\right\}N(\nu) = (k_2+\epsilon_2)N(\nu)\end{aligned}\right\} \qquad (164)$$

$$\tfrac{1}{2}m\omega^2 \equiv -E : \text{positive for bound states}$$

These equations place us in position to make formal use of the familiar quantum theory of oscillators; immediately

$$\begin{aligned}k_1+\epsilon_1 &= \hbar\omega(n_1+\tfrac{1}{2}) \quad : \quad n_1 = 0,1,2,\ldots\\ k_2+\epsilon_2 &= \hbar\omega(n_2+\tfrac{1}{2}) \quad : \quad n_2 = 0,1,2,\ldots\\ &\downarrow\\ 2k &= \hbar\omega(n_1+n_2+1) \quad \text{giving} \quad E = -4\tfrac{mk^2}{2\hbar^2}\tfrac{1}{(n_1+n_2+1)^2}\end{aligned} \qquad (165)$$

The associated (unnormalized) eigenfunctions (see Schiff's §13) are

$$\begin{aligned}M_{n_1}(\mu) &= e^{-\frac{1}{2}(\alpha\mu)^2}H_{n_1}(\alpha\mu)\\ N_{n_2}(\nu) &= \text{similar}\end{aligned} \qquad (166)$$

where

$$\alpha \equiv \sqrt{\tfrac{m\omega}{\hbar}} = \left[-\tfrac{2mE}{\hbar^2}\right]^{\frac{1}{4}} = \tfrac{1}{\sqrt{a_0}}\left[\tfrac{2}{n_1+n_2+1}\right]^{\frac{1}{2}} \qquad (167)$$

has (as required) the dimensionality of $(\text{length})^{-\frac{1}{2}}$ and—in stark contrast to the situation in oscillator theory—a meaning which is specific to each energy eigenspace.

Thus the "harmonic oscillator trick," which inspires these comments:

The hatted coordinates $\{\hat{\mu},\hat{\nu}\}$, which served so well in the 3-dimensional problem, are ill-adapted to the associated 3-dimensional theory. I am satisfied that the oscillator trick works *only* in unhatted coordinates, and only in two dimensions, but will not belabor the point.

At (165) were were led to a spectral formula

$$E = -4E_0\tfrac{1}{(\text{integer})^2}$$

which—while it does display the correct spectral depression factor—is yet at variance from the

$$E = -4E_0\tfrac{1}{(\text{odd integer})^2}$$

achieved by Zaslow & Zandler (see again p. 67). Cisneros & McIntosh[54] have identified the source of the discrepency, which I now describe. It is an obvious property of the $\{\mu,\nu\}$ coordinate system (27) that

$$\{\mu,\nu\} \text{ and } \{-\mu,-\nu\} \text{ refer to the same point } \{x,y\}$$

and it is to remove that element of redundancy that one standardly restrains the range of (say) μ, writing $0 \leqslant \mu < +\infty$, $-\infty < \nu < +\infty$.[64] We were led by the oscillator trick to hydrogenic wave functions of the form

$$\Psi(\mu,\nu) = e^{-\frac{1}{2}\alpha^2(\mu^2+\nu^2)} H_{n_1}(\alpha\mu) H_{n_2}(\alpha\nu)$$

It is, in this context, most convenient not to restrain μ but to impose

$$\Psi(\mu,\nu) = \Psi(-\mu,-\nu)$$

as an explicit condition (single-valuedness condition). And that—by a familiar parity property of the Hermite polynomials—requires that n_1 and n_2 must be either both even or both odd, which in either case entails that

$$n_1 + n_2 + 1 \text{ must necessarily be odd}$$

Half of the wave functions supplied by the oscillator trick must therefore be discarded, and these, as Cisnero & McIntosh emphasize, are precisely the states required to express the distinction between $SO(2)$ (the known dynamical group of the 2-dimensional oscillator) and O(3) (the advertised dynamical group of the 2-dimensional Kepler problem).

We were led (see again the penultimate paragraph of §13) by polar analysis to unnormalized eigenfunctions $F_{j,\ell}(r,\theta)$ which when expressed in terms of parabolic coordinates become (here I lapse into the lingo of *Mathematica*)

$$\begin{aligned} G_{j,\ell}(\mu,\nu) = \mathrm{Exp}\Big[-\tfrac{\mu^2+\nu^2}{(2j+1)a_0}\Big] &* \Big(\tfrac{2(\mu^2+\nu^2)}{(2j+1)a_0}\Big)^{\mathrm{Abs}[\ell]} \\ &* \mathrm{LaguerreL}\Big[j-\mathrm{Abs}[\ell], 2\,\mathrm{Abs}[\ell], \tfrac{2(\mu^2+\nu^2)}{(2j+1)a_0}\Big] \\ &* \mathrm{ComplexExpand}\Big[\big(\tfrac{\mu^2-\nu^2}{\mu^2+\nu^2} + i\tfrac{2\mu\nu}{\mu^2+\nu^2}\big)^{\ell}\Big] \end{aligned}$$

giving (if I allow myself temporarily to write a in place of a_0)

$$G_{0,0}(\mu,\nu) = e^{-\frac{\mu^2+\nu^2}{a}}$$

$$\begin{aligned} G_{1,+1}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} \big[2(\mu^2-\nu^2) + 4i\mu\nu\big] \\ G_{1,\ 0}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} \big[3a - 2(\mu^2+\nu^2)\big] \\ G_{1,-1}(\mu,\nu) &= \text{conjugate of } G_{1,+1}(\mu,\nu) \end{aligned}$$

[64] In the 3-dimensional case one on the other hand writes $\{\mu,\nu\} \in [0,\infty]$; see pp. 17 & 34 of Moon & Spencer.

$$
\begin{aligned}
G_{2,+2}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}\left[(4\mu^4-24\mu^2\nu^2+4\nu^4)+i(16\mu^3\nu-16\mu\nu^3)\right]\\
G_{2,+1}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}\left[(4\mu^4-30a\mu^2+30a\nu^2-4\nu^4)\right.\\
&\qquad\left.+i(8\mu^3\nu-60a\mu\nu+8\mu\nu^3)\right]\\
G_{2,\ 0}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}\left[2(\mu^2+\nu^2)^2-20a(\mu^2+\nu^2)+25a^2\right]\\
G_{2,-1}(\mu,\nu) &= \text{conjugate of } G_{2,+1}(\mu,\nu)\\
G_{2,-2}(\mu,\nu) &= \text{conjugate of } G_{2,+2}(\mu,\nu)
\end{aligned}
$$

Parabolic analysis led, on the other hand, to unnormalized eigenfunctions of the form

$$
\begin{aligned}
H_{n_1,n_2}(\mu,\nu) = \mathrm{Exp}\Big[-\tfrac{\mu^2+\nu^2}{(n_1+n_2+1)a_0}\Big] &* \mathrm{HermiteH}\Big[n_1,\sqrt{\tfrac{2\mu^2}{(n_1+n_2+1)a_0}}\Big]\\
&* \mathrm{HermiteH}\Big[n_2,\sqrt{\tfrac{2\nu^2}{(n_1+n_2+1)a_0}}\Big]
\end{aligned}
$$

giving

$$H_{0,0}(\mu,\nu) = e^{-\frac{\mu^2+\nu^2}{a}}$$

$$
\begin{aligned}
H_{2,0}(\mu,\nu) &= \tfrac{1}{3a}e^{-\frac{\mu^2+\nu^2}{3a}}(8\mu^2-6a)\\
H_{1,1}(\mu,\nu) &= \tfrac{1}{3a}e^{-\frac{\mu^2+\nu^2}{3a}}(8\mu\nu)\\
H_{0,2}(\mu,\nu) &= \tfrac{1}{3a}e^{-\frac{\mu^2+\nu^2}{3a}}(8\nu^2-6a)
\end{aligned}
$$

$$
\begin{aligned}
H_{4,0}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}(64\mu^4-480a\mu^2+300a^2)\\
H_{3,1}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}(64\mu^3\nu-240a\mu\nu)\\
H_{2,2}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}(64\mu^2\nu^2-80a\mu^2-80a\nu^2+100a^2)\\
H_{1,3}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}(64\mu\nu^3-240a\mu\nu)\\
H_{0,4}(\mu,\nu) &= \tfrac{1}{25a^2}e^{-\frac{\mu^2+\nu^2}{5a}}(64\nu^4-480a\nu^2+300a^2)
\end{aligned}
$$

By inspection

$$G_{0,0} = H_{0,0} \tag{168.0}$$

$$
\begin{aligned}
G_{1,+1} &= \ \ \tfrac{1}{4}(H_{2,0}-H_{0,2})+i\tfrac{1}{2}H_{1,1}\\
G_{1,\ 0} &= -\tfrac{1}{2}(H_{2,0}+H_{0,2})\\
G_{1,-1} &= \ \ \tfrac{1}{4}(H_{2,0}-H_{0,2})-i\tfrac{1}{2}H_{1,1}
\end{aligned}
\tag{168.1}
$$

$$\begin{aligned}
G_{2,+2} &= \tfrac{1}{16}(H_{4,0}+H_{0,4}) - \tfrac{3}{8}H_{2,2} + i\tfrac{1}{4}(H_{3,1}-H_{1,3})\\
G_{2,+1} &= \tfrac{1}{16}(H_{4,0}-H_{0,4}) + i\tfrac{1}{8}(H_{3,1}+H_{1,3})\\
G_{2,\ 0} &= \tfrac{1}{32}(H_{4,0}+H_{0,4}) + \tfrac{1}{16}H_{2,2}\\
G_{2,-1} &= \tfrac{1}{16}(H_{4,0}-H_{0,4}) - i\tfrac{1}{8}(H_{3,1}+H_{1,3})\\
G_{2,-2} &= \tfrac{1}{16}(H_{4,0}+H_{0,4}) - \tfrac{3}{8}H_{2,2} - i\tfrac{1}{4}(H_{3,1}-H_{1,3})
\end{aligned} \tag{168.2}$$

The short of it is this: polar analysis and parabolic analysis erect distinct bases within each of the respective energy eigenspaces. Those bases are interrelated by linear transformations (168) which, in the absence of normalization, we cannot expect to be unitary, but which are of a form

$$\text{(exponential)}\cdot\text{(power)}\cdot\text{(Laguerre)} = \sum \text{(gaussian)}\cdot\text{(Hermite)}\cdot\text{(Hermite)}$$

frequently encountered in the handbooks of higher analysis. Polar analysis yields basis functions which speak of angular momentum: working from

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\mu^2-\nu^2)\\ y &= \mu\nu \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \tfrac{\partial}{\partial x} &= \tfrac{1}{\mu^2+\nu^2}\big[\mu\tfrac{\partial}{\partial\mu} - \nu\tfrac{\partial}{\partial\nu}\big]\\ \tfrac{\partial}{\partial y} &= \tfrac{1}{\mu^2+\nu^2}\big[\nu\tfrac{\partial}{\partial\mu} + \mu\tfrac{\partial}{\partial\nu}\big] \end{aligned}\right. \tag{169}$$

we have

$$\mathsf{L} \sim x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x} = \tfrac{1}{2}\big[\mu\tfrac{\partial}{\partial\nu} - \nu\tfrac{\partial}{\partial\mu}\big] \tag{170}$$

and are not surprised to discover by calculation (what in polar coordinates was familiar/obvious; namely) that

$$\tfrac{1}{2}\big[\mu\tfrac{\partial}{\partial\nu} - \nu\tfrac{\partial}{\partial\mu}\big]G_{j,\ell}(\mu,\nu) = \ell\cdot G_{j,\ell}(\mu,\nu) \quad : \quad \ell = 0, \pm1, \ldots, \pm j$$

What, in the same spirit, can one say of the eigenfunctions $H_{n_1,n_2}(\mu,\nu)$?

We saw already in §5—compare (51) with (48)—that Liouville separation leads in parabolic coordinates to the conserved Lenz vector with the same naturalness that in (alternate) polar coordinates it leads to the conservation of angular momentum. Working from (137) with the aid of (169) we compute

$$\begin{aligned}
\mathsf{K}_x &= k\Big\{a_0\Big[-x\big(\tfrac{\partial}{\partial y}\big)^2 + y\tfrac{\partial}{\partial x}\tfrac{\partial}{\partial y} + \tfrac{1}{2}\tfrac{\partial}{\partial x}\Big] - \tfrac{x}{\sqrt{x^2+y^2}}\Big\}\\
&= k\tfrac{1}{2}\tfrac{1}{\mu^2+\nu^2}\Big\{a_0\Big[\nu^2\big(\tfrac{\partial}{\partial\mu}\big)^2 - \mu^2\big(\tfrac{\partial}{\partial\nu}\big)^2\Big] - 2(\mu^2-\nu^2)\Big\} \qquad (171.1)\\
\mathsf{K}_y &= k\Big\{a_0\Big[-y\big(\tfrac{\partial}{\partial x}\big)^2 + x\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x} + \tfrac{1}{2}\tfrac{\partial}{\partial y}\Big] - \tfrac{y}{\sqrt{x^2+y^2}}\Big\}\\
&= k\tfrac{1}{2}\tfrac{1}{\mu^2+\nu^2}\Big\{a_0\Big[-\mu\nu\big(\tfrac{\partial}{\partial\mu}\big)^2 + (\mu^2+\nu^2)\tfrac{\partial}{\partial\mu}\tfrac{\partial}{\partial\nu} - \mu\nu\big(\tfrac{\partial}{\partial\nu}\big)^2\Big] - 4\mu\nu\Big\} \qquad (171.2)
\end{aligned}$$

where ∇^2 has the meaning stated at (108.4). Cisneros & McIntosh (whose x and y are the reverse of my own) draw attention to alternative formulation of

the same result

$$\mathsf{K}_x = k\Big\{ -\tfrac{\mu^2-\nu^2}{2}\Big[\tfrac{1}{2}a_0\nabla^2 + \tfrac{2}{\mu^2+\nu^2}\Big] + \tfrac{1}{4}a_0\Big[\big(\tfrac{\partial}{\partial\mu}\big)^2 - \big(\tfrac{\partial}{\partial\nu}\big)^2\Big]\Big\} \qquad (172.1)$$

$$\mathsf{K}_y = k\Big\{ -\quad \mu\nu\Big[\tfrac{1}{2}a_0\nabla^2 + \tfrac{2}{\mu^2+\nu^2}\Big] + \tfrac{1}{2}a_0\tfrac{\partial}{\partial\mu}\tfrac{\partial}{\partial\nu}\Big\} \qquad (172.2)$$

which is made attractive by the observation that

$$\Big[\tfrac{1}{2}a_0\nabla^2 + \tfrac{2}{\mu^2+\nu^2}\Big] = -\tfrac{1}{k}\mathsf{H} \qquad (173)$$

Calculation proceeding from (173) gives $\mathsf{H}\,H_{0,0} = -E_0 H_{0,0}$

$$\mathsf{H}\begin{pmatrix} H_{2,0} \\ H_{1,1} \\ H_{0,2} \end{pmatrix} = -\ \tfrac{1}{9}E_0 \begin{pmatrix} H_{2,0} \\ H_{1,1} \\ H_{0,2} \end{pmatrix}$$

$$\mathsf{H}\begin{pmatrix} H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix} = -\tfrac{1}{25}E_0 \begin{pmatrix} H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix}$$

with $E_0 = 2k/a_0 = 4\cdot mk^2/2\hbar^2$, which is reassuring, but entirely expected. More interesting are the following results computed on the basis of (172):

$$\mathsf{K}_x\begin{pmatrix} H_{0,0} \\ \\ H_{2,0} \\ H_{1,1} \\ H_{0,2} \\ \\ H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ -\frac{2}{3}H_{1,1} \\ -\frac{1}{3}(H_{2,0}+H_{0,2}) \\ -\frac{2}{3}H_{1,1} \\ \\ -\frac{4}{5}H_{3,1} \\ -\frac{1}{5}(3H_{2,2}+H_{4,0}) \\ -\frac{2}{5}(\ H_{3,1}+H_{1,3}) \\ -\frac{1}{5}(3H_{2,2}+H_{0,4}) \\ -\frac{4}{5}H_{1,3} \end{pmatrix} \qquad \mathsf{K}_y\begin{pmatrix} H_{0,0} \\ \\ H_{2,0} \\ H_{1,1} \\ H_{0,2} \\ \\ H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ -\frac{2}{3}H_{2,0} \\ 0 \\ +\frac{2}{3}H_{0,2} \\ \\ -\frac{4}{5}H_{4,0} \\ -\frac{2}{5}H_{3,1} \\ 0 \\ +\frac{2}{5}H_{1,3} \\ +\frac{4}{5}H_{0,4} \end{pmatrix}$$

We are brought thus to this pretty conclusion:

- polar separation yields simultaneous eigenfunctions $G_{j,\ell}$ of H and L;
- parabolic separation yields simultaneous eigenfunctions H_{n_1,n_2} of H and K_y.

The only surprise here is that we have encountered K_y where—for no good reason, when you think about it—we might have expected K_x.

19. Keplerean contact with the algebraic theory of isotropic oscillators. It is a familiar fact that the oscillator Hamiltonian can—usefully—be written

$$H = \tfrac{1}{2m}p^2 + \tfrac{1}{2}m\omega^2 x^2 = \hbar\omega\, a^* a$$
$$a \equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[p + im\omega x\big]$$

where $\hbar$ is for the moment understood to be a constant of arbitrary value but of prescribed dimensionality, and has been introduced to render a dimensionless. For an isotropic 2-dimensional oscillator one can proceed similarly, writing

$$H = \tfrac{1}{2m}(p^2 + q^2) + \tfrac{1}{2}m\omega^2(x^2 + y^2) = \hbar\omega\,(a^* a + b^* b)$$
$$a \equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[p + im\omega x\big]$$
$$b \equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[q + im\omega y\big]$$

From the primitive Poisson bracket relations

$$[\,x\,,p\,] = [\,y\,,q\,] = 1 \quad : \quad \text{other brackets vanish}$$

follow the statements

$$[\,a\,,a^*\,] = [\,b\,,b^*\,] = i/\hbar \quad : \quad \text{other brackets vanish}$$

while from the equations of motion

$$\dot{x} = p/m \qquad \dot{y} = q/m$$
$$\dot{p} = -m\omega^2 x \qquad \dot{q} = -m\omega^2 y$$

follow

$$\dot{a} \;= +i\omega a \qquad \dot{b} \;= +i\omega b$$
$$\dot{a}^* = -i\omega a^* \qquad \dot{b}^* = -i\omega b^*$$

The construction

$$H = \hbar\omega(a^* a + b^* b)$$

is maniafestly real, and obvious constant of the motion (it refers, of course, to the conserved energy of the system). To complete the list of real quadratic constructs we form

$$D \equiv\; \hbar\omega(a^* a - b^* b) = \tfrac{1}{2m}(p^2 - q^2) + \tfrac{1}{2}m\omega^2(x^2 - y^2)$$
$$K \equiv\; \hbar\omega(a^* b + b^* a) = \tfrac{1}{m}pq + m\omega^2 xy$$
$$L \equiv i\hbar\omega(a^* b - b^* a) = \omega(xq - yp)$$

It is evident by inspection that

$$\dot{D} = \dot{K} = \dot{L} = 0 \quad : \quad D,\, K \text{ and } L \text{ are constants of motion}$$

and easy to verify that

$$H^2 = D^2 + K^2 + L^2$$

Finally, one has—in addition to $[\,H, D\,] = [\,H, K\,] = [\,H, L\,] = 0$—the Poisson bracket relations

$$\begin{aligned}[\,D, K\,] &= (2/\hbar)\,L\\ [\,K, L\,] &= (2/\hbar)D\\ [\,L\,, D\,] &= (2/\hbar)K\end{aligned}$$

I turn now from the classical mechanics to the quantum mechanics of the isotropic oscillator (in which context it was, I believe, Dirac who first drew attention to the utility of the algebraic method latent in preceding remarks). We construct non-hermitian operators

$$\begin{aligned}\mathsf{a} &\equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[\,\mathsf{p} + im\omega\,\mathsf{x}\,\big] \quad \text{and} \quad \mathsf{a}^+ \equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[\,\mathsf{p} - im\omega\,\mathsf{x}\,\big]\\ \mathsf{b} &\equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[\,\mathsf{q} + im\omega\,\mathsf{y}\,\big] \quad \text{and} \quad \mathsf{b}^+ \equiv \tfrac{1}{\sqrt{2\hbar m\omega}}\big[\,\mathsf{q} - im\omega\,\mathsf{y}\,\big]\end{aligned}$$

and from

$$[\,\mathsf{x}\,,\mathsf{p}\,] = [\,\mathsf{y}\,,\mathsf{q}\,] = i\hbar\,\mathsf{I} \quad : \quad \text{other primitive commutators vanish}$$

(note that $\hbar$ has a this point acquired a physically determined value) obtain

$$[\,\mathsf{a}^+, \mathsf{a}\,] = [\,\mathsf{b}^+, \mathsf{b}\,] = \mathsf{I} \quad : \quad \text{other commutators vanish}$$

From

$$\begin{aligned}\mathsf{a}^+\mathsf{a} &= \tfrac{1}{2\hbar m\omega}\big[\,\mathsf{p} - im\omega\,\mathsf{x}\,\big]\big[\,\mathsf{p} + im\omega\,\mathsf{x}\,\big]\\ &= \tfrac{1}{2\hbar m\omega}\big\{\mathsf{p}^2 + m^2\omega^2\,\mathsf{x}^2 - im\omega[\,\mathsf{x}\,,\mathsf{p}\,]\big\}\end{aligned}$$

we obtain

$$\begin{aligned}\tfrac{1}{2m}\big\{\mathsf{p}^2 + m^2\omega^2\,\mathsf{x}^2\big\} &= \hbar\omega\big\{\mathsf{a}^+\mathsf{a} - \tfrac{1}{2}\mathsf{I}\,\big\}\\ &= \hbar\omega\big\{\mathsf{a}\,\mathsf{a}^+ + \tfrac{1}{2}\mathsf{I}\,\big\}\end{aligned}$$

and conclude that in the 2-dimensional case

$$\begin{aligned}\mathsf{H} &= \hbar\omega\big\{\mathsf{a}^+\,\mathsf{a} + \mathsf{b}^+\,\mathsf{b} - \mathsf{I}\big\}\\ &= \hbar\omega\big\{\mathsf{a}\,\mathsf{a}^+ + \mathsf{b}\,\mathsf{b}^+ + \mathsf{I}\big\}\end{aligned}$$

The quadratic construction

$$\mathsf{N} \equiv\ (\mathsf{a}^+\,\mathsf{a} + \mathsf{b}^+\,\mathsf{b}) \equiv \mathsf{N}_1 + \mathsf{N}_2$$

is manifestly hermitian (observable), as so also are $\mathsf{N}_1 \equiv \mathsf{a}\,\mathsf{a}^+$ and $\mathsf{N}_2 \equiv \mathsf{b}\,\mathsf{b}^+$. To complete the list of such (dimensionless) constructions, we form

$$\begin{aligned}\mathsf{D} &\equiv\ (\mathsf{a}^+\,\mathsf{a} - \mathsf{b}^+\,\mathsf{b}) \equiv \mathsf{N}_1 - \mathsf{N}_2\\ \mathsf{K} &\equiv\ (\mathsf{a}^+\,\mathsf{b} + \mathsf{b}^+\,\mathsf{a}) \equiv\ (\mathsf{M}_1 + \mathsf{M}_2) = \tfrac{1}{\hbar}\big\{\tfrac{1}{m\omega}\mathsf{p}\,\mathsf{q} + m\omega\,\mathsf{x}\,\mathsf{y}\big\}\\ \mathsf{L} &\equiv i(\mathsf{a}^+\,\mathsf{b} - \mathsf{b}^+\,\mathsf{a}) \equiv i(\mathsf{M}_1 - \mathsf{M}_2) = \tfrac{1}{\hbar}(\mathsf{x}\,\mathsf{q} - \mathsf{y}\,\mathsf{p})\end{aligned}$$

Alternatively we might write

$$\mathsf{N} = \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix}^+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix} \qquad \mathsf{D} = \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix}^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix}$$

$$\mathsf{K} = \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix}^+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix} \qquad \mathsf{L} = \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix}^+ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \mathsf{a} \\ \mathsf{b} \end{pmatrix}$$

Computation[65] gives

$$[\,\mathsf{N}\,,\mathsf{D}\,] = [\,\mathsf{N}\,,\mathsf{K}\,] = [\,\mathsf{N}\,,\mathsf{L}\,] = \mathsf{O}$$

and

$$\begin{aligned} [\,\mathsf{D}\,,\mathsf{K}\,] &= 2i\,\mathsf{L} \\ [\,\mathsf{K}\,,\,\mathsf{L}\,] &= 2i\,\mathsf{D} \\ [\,\mathsf{L}\,,\mathsf{D}\,] &= 2i\,\mathsf{K} \end{aligned}$$

The latter equations mimic (to within an incidental sign) the commutation relations satisfied by the traceless hermitian matrices

$$\mathbb{D} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbb{K} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbb{L} \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

which we recognize to be "Pauli matrices"—generators of the fundamental representation of $SU(2)$. Additional computation[66] gives

$$\mathsf{D}^2 + \mathsf{K}^2 + \mathsf{L}^2 = \mathsf{N}\,(\,\mathsf{N} - \mathsf{2}\,)$$

The non-hermitian operators a, b and a^+, b^+ become up/down "ladder operators"—and the hermitian operators N_1 and N_2 become at the same time "number opertors"—in consequence of the elementary facts summarized below:

$$\begin{aligned} \mathsf{N}_1\,\mathsf{a} &= \mathsf{a}\;\;(\,\mathsf{N}_1 + \mathsf{I}\,) & \qquad \mathsf{N}_2\,\mathsf{a} &= \mathsf{a}\;\;\mathsf{N}_2 \\ \mathsf{N}_1\,\mathsf{a}^+ &= \mathsf{a}^+\,(\,\mathsf{N}_1 - \mathsf{I}\,) & \qquad \mathsf{N}_2\,\mathsf{a}^+ &= \mathsf{a}^+\;\;\mathsf{N}_2 \\ \mathsf{N}_1\,\mathsf{b} &= \mathsf{b}\;\;\mathsf{N}_1 & \qquad \mathsf{N}_2\,\mathsf{b} &= \mathsf{b}\;\;(\,\mathsf{N}_2 + \mathsf{I}\,) \\ \mathsf{N}_1\,\mathsf{b}^+ &= \mathsf{b}^+\;\;\mathsf{N}_1 & \qquad \mathsf{N}_2\,\mathsf{b}^+ &= \mathsf{b}^+\,(\,\mathsf{N}_2 - \mathsf{I}\,) \end{aligned}$$

For if $|\psi)$ is an eigenstate of N_1 then so also is $\mathsf{a}|\psi)$, with eigenvalue incremented by unity:

$$\mathsf{N}_1|\psi) = \lambda\,|\psi) \quad \Longrightarrow \quad \begin{cases} \mathsf{N}_1\mathsf{a}\;\;|\psi) = (\lambda+1)\,\mathsf{a}\;\;|\psi) \\ \mathsf{N}_1\mathsf{a}^+|\psi) = (\lambda-1)\,\mathsf{a}^+|\psi) \end{cases} \quad \text{etc.}$$

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[65] It is simplest to proceed from the observation that

$$\begin{aligned} [\,\mathsf{N}_1\,,\mathsf{N}_2\,] &= \mathsf{O} & \qquad [\,\mathsf{N}_1\,,\mathsf{M}_1\,] &= -\mathsf{M}_1 & \qquad [\,\mathsf{N}_1\,,\mathsf{M}_2\,] &= +\mathsf{M}_2 \\ & & \qquad [\,\mathsf{N}_2\,,\mathsf{M}_1\,] &= +\mathsf{M}_1 & \qquad [\,\mathsf{N}_2\,,\mathsf{M}_2\,] &= -\mathsf{M}_2 \\ & & & & \qquad [\,\mathsf{M}_1\,,\mathsf{M}_2\,] &= \mathsf{N}_2 - \mathsf{N}_1 \end{aligned}$$

[66] Easily $\mathsf{D}^2+\mathsf{K}^2+\mathsf{L}^2 = \mathsf{N}_1^2 - 2\mathsf{N}_1\mathsf{N}_2 + \mathsf{N}_2^2 + 2(2\,\mathsf{M}_1\mathsf{M}_2 + \mathsf{N}_1 - \mathsf{N}_2)$. The argument is completed by the observation that $\mathsf{M}_1\mathsf{M}_2 = \mathsf{N}_1\mathsf{N}_2 - \mathsf{N}_1$.

In simple oscillator theory one constructs the ground state $|0)$ by imposing the requirement that $\mathsf{a}^+|0) = 0$, which in the x-representation reads

$$\tfrac{d}{dx}\psi_0(x) = -\tfrac{m\omega}{\hbar}x\,\psi_0(x)$$

and entails

$$\psi_0(x) \sim e^{-\frac{1}{2}(\beta x)^2} \quad \text{with } \beta \equiv \sqrt{m\omega/\hbar}$$

Normalized excited states are then constructed[67]

$$|n) = \tfrac{1}{\sqrt{n!}}\,\mathsf{a}^n\,|0)$$

giving

$$\psi_n(x) \equiv (x|n) \sim e^{-\frac{1}{2}(\beta x)^2} H_n(\beta x)$$

Returning now to the isotropic oscillator, one is led to normalized eigenstates of N —equivalently of H —which can be described

$$\begin{aligned}\psi_{n_1 n_2}(x,y) &\equiv (x,y|\tfrac{1}{\sqrt{n_1!n_2!}}\,\mathsf{a}^{n_1}\mathsf{b}^{n_2}\,|0,0)\\ &\sim e^{-\frac{1}{2}\beta^2(x^2+y^2)} H_{n_1}(\beta x) H_{n_2}(\beta y)\end{aligned}$$

and where the "bottom state" (physicall the ground state) is defined by the properties

$$\mathsf{a}^+\,|0,0) = \mathsf{b}^+\,|0,0) = 0$$

One has

$$\begin{aligned}\mathsf{N}_1\,|n_1,n_2) &= n_1\,|n_1,n_2)\\ \mathsf{N}_2\,|n_1,n_2) &= n_2\,|n_1,n_2)\end{aligned}$$

whence

$$\begin{aligned}\mathsf{N}\ \ |n_1,n_2) &= n\ \ |n_1,n_2)\\ n &= n_1+n_2 = 0,1,2,\ldots\end{aligned}$$

from which we infer that the n^{th} eigenvalue of N is $(n+1)$-fold degenerate. The non-hermitian operators M_1 and M_2 (which evidently[65] commute with N) acquire in this light new interest; from

$$\mathsf{M}_1\,|n_1,n_2) \sim \begin{cases} |n_1-1,n_2+1) & \text{if } n_1 = 1,2,3,\ldots\\ 0 & \text{if } n_1 = 0\end{cases}$$

$$\mathsf{M}_2\,|n_1,n_2) \sim \begin{cases} |n_1+1,n_2-1) & \text{if } n_2 = 1,2,3,\ldots\\ 0 & \text{if } n_2 = 0\end{cases}$$

we see that M_1 and M_2 act to move one $\downarrow$ and $\uparrow$ through the stack of states

[67] See QUANTUM MECHANICS (1967), Chapter 2, pp. 56–59 for details, but beware of

$$
\begin{aligned}
&|n \quad\;\; , 0) \\
&|n-1, 1) \\
&|n-2, 2) \\
&\vdots \\
&|2, n-2) \\
&|1, n-1) \\
&|0, n \quad\;\;)
\end{aligned}
$$

which span the n^{th} eigenspace of N, and within which $\{\mathsf{D}, \mathsf{K}, \mathsf{L}\}$ act to generate an $(n+1)$-dimensional representation of $SU(2)$.

All of which has clear relevance to material that emerged from the parabolic formulation of the quantum mechanics of the 2-dimensional Kepler problem, though two fundamental distinctions have to be made:

- The formal substitution $\{x, y\} \mapsto \{\mu, \nu\}$ was found (p. 77) to entail that we enforce the single-valuedness condition $\Psi(\mu, \nu) = \Psi(-\mu, -\nu)$, and thus to require that *in application to the Kepler problem we admit only even values of* n;
- The formal substitution

  $$\beta \equiv \sqrt{\tfrac{m\omega}{\hbar}} \quad \longmapsto \quad \alpha \equiv \sqrt{\tfrac{2mk}{\hbar^2(n+1)}}$$

  brings into play an *n-dependent scale factor which varies from eigenspace to eigenspace*.[68] Bohr's spectral formula (118/144) yields this alternative statement

  $$m\omega \quad \longmapsto$$

The latter circumstance bedevils the design of the "Keplerean ladder operators," in ways which Cisneros & McIntosh have labored to disentangle, but is relatively benign as relates to operators (like N_1, N_2, M_1 and M_2, and operators assembled from them) which return eigenstates to the eigenspaces in which they originated. The force of the latter assertion is illustrated by the remarks which bring us now to the point of this discussion:

In oscillator theory one has

$$\mathsf{N} = \mathsf{N}_1 + \mathsf{N}_2 =$$

[68] Note that the substitutions $\mapsto$ involve replacing a variable/parameter by another *of different physical dimension*.

I conclude this discussion with some historical remarks. The precocious Wolfgang Pauli (1900–1958) had, at quite an early point in his career, become a central figure in the effort to use resources of the "old quantum theory" (Bohr–Sommerfeld) to deepen the theory of atomic spectra, and in particular to account for the "anomalous Zeeman effect;"[69] it was that involvement—and the appearance in 1924 of a seminal paper by E. C. Stoner—which led him in 1925 to formulation of the "Pauli exclusion principle,"[70] and on Heisenberg's heels (in advance of Schrödinger) to the argument summarized in §14. But two events—the invention by Krönig and by Uhlenbeck & Goudsmit (1925/26) of the concept of electron "spin;" the invention of the Schrödinger equation (1926) and of the collateral notion of a "multi-component wave function" (X. Darwin, 1927)—necessarily preceded the formulation[71] of "Pauli's non-relativistic theory of electron spin," which (though immediately superceded by Dirac's relativistic theory as it was destined to be) marked a major milestone toward understanding of the *physical origin* of the exclusion principle, and to which we commonly look to find the first occurance of the "Pauli spin matrices." The point to which I would draw attention is that, as I have recently had occasion to emphasize, Pauli matrices and all the associated $SU(2)$ apparatus were present already at the algebraic heart of Pauli's account (§14) of "das Waßerstgoffspektrum"[43] (hydrogen atom with *spinless electrons*); the (re-)occurance of Pauli matrices in the "quantum theory of (intrinsic) angular momentum" is a later development. Here once again, the historical trend has been from (what we are today inclined to regard as) the obscurely intricate to the simple, from the arcane to the commonplace—the reverse of what one might naively have imagined it to be. I would point out also that the mathematical essentials of the "$SU(2)$ theory of isotropic oscillators (whence of 2-dimensional hydrogen)" had been sketched by George Stokes (1850's) and elaborated by Henri Poincaré (1890's) many years before Pauli entered the picture, and in connection with an "elliptic orbits problem" presented by physics of quite a different nature—the physics of optical polarization.[72]

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[69] His "Über die Gesetzmäßigkeiten des anomalen Zeemaneffekts" appeared already in 1923 (Z. Physik **16**, 155).

[70] "Über den Zusammenhang des Abschoßes der Elektronengruppen im Atom mit der Komplexstruktur," Z. Physik **31**, 765. English translations of that paper and of Stoner's can be found in I. Duck & E. C. G. Sudarshan, *Pauli and the Spin-Statistics Theorem*(1997). For a careful account of the developments feeding into and radiating from this nodal point in the history of physics, see Max Jammer, *Conceptual Development of quantum Mechanics* (1966).

[71] "Zur Quantenmechanik des magnetischen Elektrons," Z. Physik **43**, 601 (1927).

[72] The details are developed in my "Ellipsometry," (1999).